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The Development of Gifted and Talented Mathematics Students and the National Council of Teachers of Mathematics Standards

Linda Jensen Sheffield
Northern Kentucky University
Highland Heights, Kentucky

June 1994
RBDM 9404

MATHEMATICS
RESEARCH-BASED DECISION MAKING SERIES

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About the Author...

Dr. Linda Jensen Sheffield, a professor with a joint appointment in the School of Education and the Department of Mathematics and Computer Science at Northern Kentucky University, has done research in the area of the development of mathematical talent for over twenty years. She has directed three Young Scholars grants from the National Science Foundation dedicated to encouraging talented middle grade students to develop their abilities in mathematics, engineering, and science and has directed several Eisenhower grants and a United States Department of Education grant designed to assist teachers in the development of teaching methods using technology and manipulatives. She is the author of a series of problem-solving books for Scholastic, three books on using mathematics manipulatives for Carson-Dellosa, and is working on the third edition of an elementary and middle school mathematics education book for Merrill. She has published articles in a variety of journals including the *Arithmetic Teacher,* the *Gifted Child Quarterly,* and *School Science and Mathematics*.

The Development of Gifted and Talented Mathematics Students and the National Council of Teachers of Mathematics Standards

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ABSTRACT

Our top students in mathematics are crucial to the well-being of our country. The only way we can meet our national goal of being first in the world in mathematics and science is to raise the mathematical competence of all our students, including the gifted and talented ones.

Currently, the top mathematics students in the United States have fallen behind those in the rest of the world. These students must be nurtured and encouraged to develop their talents. The National Council of Teachers of Mathematics (NCTM) has stated in their position paper on *Provisions for Mathematically Talented and Gifted Students* that "while all students need curricula that develop the students' problem solving, reasoning, and communication abilities, the mathematically talented and gifted need in-depth and expanded curricula that emphasize higher order thinking skills, nontraditional topics, and the application of skills and concepts in a variety of contexts" (NCTM, 1993). In 1989, NCTM developed the *Curriculum and Evaluation Standards for School Mathematics* as guidelines for improving the mathematical competence of all our students. This was followed in 1991 by the *Professional Standards for Teaching Mathematics*, a set of guidelines designed to help teachers create an environment in which all students can develop mathematical power. In 1993, a working draft of a third document, *Assessment Standards for School Mathematics,* was developed to expand and complement the Evaluation Standards that were included in the 1989 document. The implications for the development of mathematical talent using all three sets of these Standards are included in this paper.

Mathematical talent must be identified through a range of measures that go beyond traditional standardized tests. Measures should include observations, student interviews, open-ended questions, portfolios, and teacher-, parent-, peer- and self-nomination. Recognition should be made of the fact that mathematical talents can be developed; they are not just something with which some students were born. Interesting tasks must be presented that engage students and encourage them to develop their mathematical talents.

Qualified mathematics teachers, improved opportunities for mathematics learning, and a much more challenging, nonrepetitive, integrated curriculum are needed to help students develop mathematical talents. Students must be challenged to create questions, to explore, and to develop mathematics that is new to them. They need outlets where they can share their discoveries with others.

Program options for the development of gifted and talented students might encompass a variety of methods including differentiated assignments, a core curriculum, pull-out programs, in-class programs, magnet schools, and extracurricular activities such as after-school or Saturday programs, mentorship programs, summer programs, and competitions.

We must act immediately on a national level to upgrade the level of mathematics being offered to all our top students from kindergarten through graduate school. Perhaps, even more importantly, we must improve the ways in which our students learn mathematics. Teachers must become facilitators of learning to encourage all students to construct new, complex mathematical concepts. Students must be challenged to reach for ever-increasing levels of mathematical understanding. We must strive to help many more students including females, minorities, and students from rural and inner-city schools reach those top levels of mathematical ability. The potential exists in every school in our country for far more expertise in mathematics, and we must help students unlock their talents in this area.

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EXECUTIVE SUMMARY

By the year 2000, U.S. students will be first in the world in science and mathematics achievement.
America 2000: An Education Strategy, 1991

In order to meet this lofty goal, we must make changes in our 110,000 public and private schools as well as in the homes and businesses of America. We must realize the importance of the development of gifted and talented students in our schools today and in the future who will be the leaders in the movement toward this goal. We are currently far from meeting this goal, and in order to reach it, changes in our attitudes, our curriculum, our teaching methods, and our means of assessment must be drastic and immediate. These changes must affect all students, including the gifted and talented.

Overview

In 1986, the Board of Directors of the National Council of Teachers of Mathematics established the Commission on Standards for School Mathematics as one means to begin these changes. This group developed a set of standards for the K-12 mathematics curricula in North American schools and a set of standards for evaluating the curricula and student achievement. The *Standards* were drafted in 1987, circulated among interested groups and revised in 1988, and published as the *Curriculum and Evaluation Standards for School Mathematics* in March, 1989. Currently, teachers, school district personnel, state level consultants, and many others across the United States and Canada are using these *Standards* as a basis for changes in their mathematics programs.

The *Curriculum and Evaluation Standards for School Mathematics* were followed by the *Professional Standards for Teaching Mathematics* in 1991. The *Professional Standards* acknowledged that teachers are the most powerful force in making changes in the ways in which mathematics is taught and learned in our schools and that they need adequate resources and long-term support in order to make these needed changes. The *Professional Standards* have numerous suggestions for ways that teachers can help students develop their mathematical talents.

The most recent of the *Standards* are the *Assessment Standards for School Mathematics*, with a working draft published in October, 1993. The *Assessment Standards* were designed to expand and complement, not replace, the Evaluation Standards that were part of the 1989 document. The *Assessment Standards* are based upon the assumption that every student is capable of achieving mathematical power (NCTM, 1993a, p. 3). This document includes a number of suggestions that can be applied to helping the top students develop even more power.

In this paper, ways to use all three sets of the *Standards* for the development of gifted and talented mathematics students are explored and discussed. Even though most areas of the three *Standards* do not directly address gifted and talented students, many of the recommendations are directly applicable. In 1993, NCTM released a draft of its position paper on *Provisions for Mathematically Talented and Gifted Students*. This paper was designed to replace an earlier position paper on gifted and talented students and includes recommendations based on the *Standards*. The position paper stated:

> It is the position of NCTM that all students can benefit from an opportunity to study the core curriculum specified in the *Standards*. This can be accomplished by expanding and enriching the curriculum to meet the needs of each individual student, including the gifted. (NCTM, 1993b)

Before looking at the recommendations from the *Standards,* research on the characteristics and identification of students displaying mathematical gifts and talents is reviewed. Students who show mathematical talent may or may not have similar gifts and talents in other areas. These students may be overlooked or underserved in the regular classroom or even in a general program for gifted students. At particular risk are females, minorities, and other traditionally underserved talented students in mathematics such as rural and inner-city students.

Because of the considerable number of recent reports on the poor showing of United States students on national and international mathematics tests, research on the level of mathematical competency of the gifted and talented students in the United States is also discussed. If the United States is going to remain a superpower in the world, we must pay more attention to our top mathematics students. We cannot be leaders in a technological world without the top schools and students in mathematical and technical fields.

An overview of the three sets of *Standards* is included along with recommendations for curriculum, teaching, and assessment of students showing mathematical talent. Specific recommendations for programs, teaching strategies, resources, and alternative means of evaluation suitable for gifted and talented mathematics students related to the *Standards* are suggested. It is hoped that teachers will take the suggestions and build upon them to meet the individual needs of their own students.

Gifted and Talented Mathematical Behaviors

Description of Behaviors

Mathematical talents are demonstrated in a variety of ways. Being a gifted or talented mathematics student has frequently been defined as scoring above the 95th percentile on a test of mathematical achievement. This is a very narrow definition of giftedness, however, and may contribute to the small number of successful mathematicians who are born and raised in the United States.

General lists of characteristics of mathematical talents include characteristics of academic gifts in general such as fast learning pace, keen observation skills, powerful questioning skills, exceptional reasoning capacity, and creativity. Those characteristics specific to mathematics usually include the following:

- Early and keen awareness, curiosity, and understanding about quantitative information
- Ability to perceive, visualize, and generalize patterns and relationships
- Ability to reason analytically, deductively, and inductively
- Ability to reverse reasoning processes, and to switch methods easily but not impulsively
- Ability to work with mathematical concepts in fluent, flexible, and creative ways
- Energy and persistence in solving difficult problems
- Ability to transfer learning to novel situations
- Tendency to formulate mathematical questions not just to answer them
- Ability to organize and work with data in a variety of ways and to disregard irrelevant data. (House, 1987; Greenes, 1981)

Notice that the list does not include the ability to compute rapidly and accurately. While some students showing mathematical talents may have this ability, it is not a necessary or sufficient characteristic of gifted mathematics students. Many gifted and talented mathematics students are impatient with details and do not care to spend time on computation. They are anxious to get on to the important aspects of the problems. Some may enjoy the challenge of being the fastest and most accurate on timed computation tests, but many are turned off by the low level of reasoning required.

Identification of Mathematical Giftedness

Not all students with mathematical talent will have all the abilities listed above. Some students may exhibit some of these characteristics spontaneously, and others may display their talents only when presented with interesting problems. Many of the talents can be developed, and all students should be given problems where they can demonstrate their talents.

Standardized achievement tests may not identify students who are gifted and talented in mathematics. One reason is that these tests often concentrate on low level tasks (Romberg & Wilson, 1992).

In spite of their limitations, achievement tests are perhaps the most widely used means of identifying mathematically talented students through the use of out-of-grade level testing. The Study of Mathematically Precocious Youth (SMPY), begun by Julian Stanley of Johns Hopkins University in 1971, identifies talented mathematics students through the use of the College Board's Scholastic Aptitude Test (SAT) or the American College Testing Program (ACT). Through this talent search, students in seventh or eighth grade who test in the top 3% on a standardized achievement test are eligible to take the ACT or SAT to qualify as mathematically precocious.

A perhaps lesser known talent search is the U.S.A. Mathematical Talent Search directed by George Berzsenyi of Rose-Hulman Institute of Technology. Unlike the SMPY Talent Search, the U.S.A. Mathematical Talent Search sends out problems to students that they then have a month or longer to complete. The search includes several rounds of problems with each round containing five problems. Students are expected to send in solutions to at least two of the problems to then compete in the next round. This talent search is more in keeping with the NCTM *Standards* in that it not only fosters insight, ingenuity, and creativity, but it also rewards perseverance by allowing students to work on problems long enough to display their mathematical abilities (Berzsenyi, 1993).

Other means of identifying mathematically talented students include observations, interviews, and self-, parent-, teacher-, and peer-identification.

If we do not provide students with the opportunity to hone their talents through practice with stimulating problems, we may well be missing a tremendous opportunity for the development of mathematical abilities in the United States. We must stimulate all students in mathematics and expect our best students to work at a level far above that currently expected.

Status of Mathematical Talent in the United States

National Comparisons

Mathematics scores on standardized tests have risen slightly during the past 15 years; however, fewer students are choosing careers in math, science, and engineering. In 1982, half of the students scoring in the top 10% on the SAT planned to major in math, science, or engineering. Only 44% had such plans in 1986. For females, this number is even more discouraging. In 1986, only about 15% of the white females in the top 10% on the SAT planned to major in a highly quantitative field (Grandy, 1987).

International Comparisons

As reported by Callahan (1992), Miwa noted that the type of items from Level 350 of the National Assessment of Educational Progress test are asked of fifth and sixth grade students in Japan. More than 60% of Japanese students under the age of 13 can answer these questions, while only .4% of American 13-year-olds and 7% of American 17-year-olds can answer these questions. The level of work that is expected of fifth and sixth grade Japanese students is not introduced in most American schools until high school and is mastered by only 7% of American high school graduates.

The fact that math scores in the United States have gone up in the past 20 years means little when we compare our test results to those in other countries. Typical math questions on the ACT and SAT require far less mastery of mathematics than similar tests in other countries (Wu, 1993). Teachers and parents do not expect nearly as much of U.S. students as do teachers and parents in the rest of the world.

Perhaps the most disturbing indicator of how our top mathematics students are doing comes from international studies. The Second International Mathematics Study (Crosswhite, et al., 1986) showed that performance of the top 5% of U.S. students in college preparatory mathematics is exceeded by 50% of the students in Japan. The International Association for the Evaluation of Educational Achievement (IEA) (McKnight, et al., 1987) showed the top 3% of American students only earned scores at the average of all students taking the same level of mathematics in other countries. Our very best students—the top 1%—scored lowest in algebra of the top 1% in all participating countries. Of the top 5%, U.S. students were above only the top 5% of students in Israel. In functions and calculus, the top 1% of U.S. students scored only a few points above students from British Columbia, Canada, even though calculus is not part of the Canadian curriculum.

Poor showings on international comparisons are not limited to high school students. Stevenson, Lee, and Stigler (1986) compared mathematics achievement of Japanese, Chinese, and American first and fifth graders. In first grade, only 15 Americans were among the top 100 scorers and by fifth grade only one American was among the top 100 scorers. Americans were found to be as inefficient at computation as they were at solving word problems.

From these and other studies, it appears that as early as first grade, U.S. students (including the very best students) perform more poorly than Asian students and this trend continues through high school and beyond. In the United States, this is apparently due to a combination of factors that include low expectation on the part of parents, teachers, and the students themselves, a belief that success in mathematics is due to ability rather than effort, a repetitive curriculum that does not expose children to new, more difficult concepts, a lack of time to reflect upon and discuss mathematical problems, and a lack of classroom instruction and homework time devoted to mathematics.

The National Council of Teachers of Mathematics

Position Paper on *Provisions for Mathematically Talented and Gifted Students*

In addition to the NCTM *Standards*, the National Council of Teachers of Mathematics has an Instructional Issues Committee that considers current issues and writes position papers on the NCTM stand on these issues. These are studied and discussed by the mathematics education community before being officially adopted by NCTM. In 1993, the Instructional Issues Committee drafted a new position paper on *Provisions for Mathematically Talented and Gifted Students*. This paper acknowledges the importance of opportunities for gifted and talented students to develop to their full potential. *Provisions for Mathematically Talented and Gifted Students* (draft) states that "while all students need curricula that develop the students' problem solving, reasoning, and communication abilities, the mathematically talented and gifted need in-depth and expanded curricula that emphasize higher order thinking skills, nontraditional topics, and the application of skills and concepts in a variety of contexts. . . Therefore, the National Council of Teachers of Mathematics recommends that all mathematically talented and gifted students have access to appropriate curricula and instruction that contributes to developing positive attitudes, furthering their mathematical interests, and encouraging their continuing participation in the study of mathematics." Appropriate curricular suggestions, instructional strategies, and assessment ideas will be considered in the following sections.

Curriculum and Evaluation Standards for School Mathematics

Overview

In 1986, the National Council of Teachers of Mathematics (1989a) appointed a Commission on Standards for School Mathematics and charged it with two tasks:

- Create a coherent vision of what it means to be mathematically literate in a world that relies on calculators and computers to carry out mathematical procedures, and in a world where mathematics is rapidly growing and is extensively being applied in diverse fields.
- Create a set of standards to guide revision of the school mathematics curriculum and associated evaluation toward this vision. (p. 5)

The *Standards* recommend curricular areas for study by all students. In the past, much of the curriculum for gifted students included areas of study that were frequently not included in the curriculum for all students such as probability and statistics. These areas are now recommended for everyone, not just the gifted. The recommended methods of teaching for all students also include techniques that were at one time restricted to gifted students such as asking students to reason and explain their thinking, using concrete models to demonstrate mathematical concepts, and making and testing hypotheses about the nature of mathematics. With these techniques now recommended for everyone, what should be done for gifted and talented students?

The *Standards* do acknowledge the gifted. The *Standards* (1989b) state:

> This, however, does not suggest that we believe all students are alike. We recognize that students exhibit different talents, abilities, achievements, needs, and interests in relationship to mathematics. The mathematical content outlined in the *Standards* is what we believe all students will need if they are to be productive citizens in the twenty-first century. If all students do not have the opportunity to learn this mathematics, we face the danger of creating an intellectual elite and a polarized society.
>
> On the one hand, prior to grade 9, we have refrained from specifying alternative instructional patterns that would be consistent with our vision. On the other hand, for grades 9-12, the standards have been prepared in light of a core program for all students, with explicit differentiation in terms of depth and breadth of treatment and the nature of applications for college-bound students . . . their experiences may differ in the vocabulary or notations used, the complexity of arguments, and so forth. (p. 9)

The *Standards* contain a warning for advanced high school students:

> A school curriculum in line with these standards should be organized so as to permit all students to progress as far into the mathematics proposed here as their achievement with the topic allows. In particular, students with exceptional mathematical talent who advance through the material more quickly than others may continue to college-level work in the mathematical sciences. However, we strongly recommend against acceleration that either omits content identified in these standards or advances through it superficially. (p. 124)

Even though the *Standards* recommend high standards for all students, that does not mean that all students are expected to reach the same levels in the same way. All students should study the same curricular topics, but some students should be expected to study the topics in greater depth, making more connections and generalizations than others. Students may want to study some topics in more depth than other topics. Just because a student displays mathematical talents does not mean that that student will want to or should be expected to master every topic studied to the same degree. The student should be expected to ask nonsuperficial questions about the topic and to explore a variety of means of answering those questions, however, for all strands of the *Standards*.

Curriculum Suggestions

Talented students should always be encouraged to think more deeply about mathematical topics being studied. As students create projects and investigations, teachers should keep copies of the very best work each year. When students see examples of outstanding work, they realize what is expected of them. They then endeavor to improve upon others' best efforts. Each year, the best work should get even better.

Any projects chosen by gifted and talented students should be related to the planned curriculum that should be taken from one of the strands of the NCTM *Standards*. Teachers should keep a file of interesting problems that they have used with students or that they find interesting to study on their own. Students and teachers alike should regularly read journals and other publications to get new ideas for further investigations.

The Core Curriculum

The NCTM Curriculum Standards recommend that all high school students study the same core curriculum in the first three years of high school. Some expanded topics are added for college-bound students in the senior year. This Core Curriculum recommends that:

> . . . differentiation in learning outcomes occurs by blending core lessons for all students with extended activities that students can complete to different depths and levels of abstraction and formalism. As should be the case with all student investigations, provisions are made for students to share their experiences, clarify their thinking. generalize their discoveries, and construct convincing arguments. (Hirsch, 1992, p. vii)

In a core curriculum, therefore, all students study a rich, nonrepetitive curriculum, with the top students being challenged by extended activities that require greater depth of exploration and generalization.

Criteria for Challenging Mathematics Curricula

In judging whether curriculum for gifted and talented students in mathematics is appropriate, it should meet the following criteria. Many of the criteria are recommended for all students, and all students should be encouraged to explore mathematical topics as deeply and thoroughly as possible.

1. The curriculum should challenge students to use and explain logical inductive and deductive reasoning.
2. Curriculum materials should encourage students to ask questions and make generalizations that go beyond the original problem.
3. Curriculum should suggest a variety of methods, materials, and technology to solve a given problem. Even the very best students should not be restricted to paper and pencil or mental problem solutions.
4. The curriculum should cover all the areas of the *Standards* recommended for that grade level. Topics such as geometry, algebra, statistics, and probability should be integrated, not studied in isolation. Many topics should be explored in greater depth by top students.
5. Measurement of achievement should include a wide range of measures such as observations, interviews, exhibitions, demonstrations, portfolios, open-ended questions, and performance events.

6. There should be opportunities for students to work with others at their achievement level as well as opportunities for students to do independent research and investigations. Bright students need the opportunity to discuss mathematical concepts with others of their developmental level.
7. Assignments should be flexible enough to allow students to demonstrate mastery of topics such as low level computation skills and allow for more extensive, deeper investigation of interesting topics.
8. Lectures and repetition should be avoided or kept to a minimum to allow students time and opportunities to be coinvestigators with the instructor and with each other.
9. Extensive use should be made of concrete, manipulative materials for students of all ages. Students should make abstract generalizations based on their concrete manipulation of materials when possible.
10. Expectations should be very high. Students should be expected to go beyond previous levels with each new task. They should make connections to previous mathematics, other subjects, and everyday life, and use elegant, precise, accurate language to describe their thinking and their results. They should be challenged to extend and generalize new learning whenever possible.

Professional Standards for Teaching Mathematics

Overview

In 1991, the *Professional Standards for Teaching Mathematics* was published as a companion document to the *Curriculum and Evaluation Standards for School Mathematics*. This document gives insight into the direction NCTM has recommended for mathematics instruction. As with the *Curriculum Standards*, the *Professional Standards* contain recommendations for teaching all students, but many of the suggestions are invaluable to teachers of our best students. The *Professional Standards* are based upon the following assumptions:

- Teachers are key figures in changing the ways in which mathematics is taught and learned in schools.
- Such changes require that teachers have long-term support and adequate resources.
- Effective teachers are those who can stimulate students to learn mathematics.
- Students learn mathematics well only when they construct their own mathematical understanding. (NCTM, 1991, p. 2)

Teachers of the gifted will recognize many of these recommended shifts as long-time practices in gifted and talented programs. These shifts are now being recommended for teachers of all students. It is noted in the *Standards*, "however, this does not mean that every child will have the same interests or capabilities in mathematics" (NCTM, 1991, p. 4). Some students will continue to perform at higher levels than others, and we must raise our expectations for these top students as well.

Teaching Heuristics and Strategies

Because the *Standards* are directed at teachers of all students, it is useful for teachers to have teaching heuristics and strategies designed specifically for the development of mathematical talent. A heuristic is a general method of solving a problem. Heuristics can help students in getting started on the solution to a problem when they might otherwise give up because they did not know where to begin.

Program Options for Gifted and Talented Mathematics Students

There are a variety of ways to provide for the needs of gifted and talented students in mathematics. A good program should include several of the following to allow students and parents to choose the means most suited to the students' individual needs: in-class programs, pull-out programs, magnet schools, and extracurricular activities such as after-school, Saturday, or summer programs, mentorship programs, and competitions.

Traits of Teachers of Gifted and Talented Mathematics Students

To properly serve the needs of students who exhibit mathematical talents and interest, teachers who understand these students and who engage in deep mathematical thinking themselves are needed. These teachers not only need to be able to diagnose the level of the students, but also must be able to suggest appropriate and challenging activities.

Several traits characterize good teachers of gifted and talented mathematics students. Many of these are traits of all good mathematics teachers, but they are especially needed by teachers of our top students. These include:

- an enthusiasm for mathematics and for teaching. Teachers need to be able to convey a sense of the beauty and wonder of mathematics.
- a confidence about their own mathematical abilities. Teachers may not know all the answers to the students' questions, but they should be unafraid to admit a lack of knowledge and to model for the students ways in which to reach an answer on their own.
- a strong mathematical background. In order to challenge students with appropriate problems, teachers must have knowledge of a variety of mathematical topics and should be actively involved in professional development in the field.
- a flexibility and a willingness to be co-investigators with the students. Students will frequently ask questions that lead the class in directions not foreseen by the teacher. Teachers should be ready and willing to follow the lead of the students as they investigate unplanned areas.

- a willingness to give up the lectern and the chalk. Gifted students need to take over the direction and responsibility for their own learning of mathematics with teachers acting as the "guide on the side."

Assessment Standards for School Mathematics

Overview

In October, 1993, NCTM released a working draft of the *Assessment Standards for School Mathematics*. These *Standards* were created to complement the two previous NCTM *Standards*. They were designed to complement, not replace, the Evaluation Standards that were included in the 1989 *Standards*.

The draft of the *Assessment Standards* are based upon the assumptions that every student is capable of achieving mathematical power, and that assessment needs to take many forms and serves a variety of purposes.

Notice that the *Assessment Standards*, like the other two NCTM *Standards*, are based upon the assumption that all students can achieve mathematical power. Again, this does not mean that all students will develop the same mathematical power.

Assessment Strategies

Assessment of mathematical power has in the past frequently been defined as the score on a standardized test of mathematics achievement. Problems with this approach have been discussed previously. Scores on traditional mathematics tests frequently stress lower level skills that a program for talented students should not be emphasizing. In order to properly judge the success of these students, the assessment needs to match the level of thinking being emphasized in the program. Raising the level of thinking in the tests can also serve to raise the level of thinking being taught in the programs. This involves assessment that uses a variety of measures to test the three areas mentioned in the *Assessment Standards*: observations, student responses to questions, and examinations of student products.

Summary of Recommendations

Mathematical Talent

In order to allow students to demonstrate that they are capable of top mathematical performance, we must do the following:

1. Give students a wide variety of rich, inviting tasks that require spatial as well as analytic abilities. Both of these are very important to success in mathematics. Good mathematicians must be skilled in both areas.

2. Encourage students to persist in solving mathematical problems. Difficult tasks require work for even the most gifted students. Many students give up before they give mathematics a chance.
3 . Expect students to not only solve problems posed by others but to pose and solve new problems of their own. If students are only asked to compute, we will never know who can perform at the top levels in mathematics.

Identification

To identify mathematical talent, the following steps are recommended:

1. We must use a variety of identification measures. Standardized tests measure only a very narrow range of generally low level skills.
2. We must provide students with assessment tasks that tap skills beyond computation. These tasks can frequently not be measured by paper and pencil multiple choice tests.
3. We must have a wide range of opportunities such as exciting mathematics classes, mathematical clubs and contests where students can demonstrate and hone their mathematical abilities.

Status of Mathematical Talent in the United States

In light of national and international studies of mathematics achievement, the following recommendations can be made:

1. Students in the United States need the opportunity to learn more mathematics. This is especially true of our top students who need the challenge of new and more complex problems, rather than the repetition of a typical mathematics curriculum.
2. Teachers and students in the United States need to tackle fewer problems, but in far greater depth. Time is needed for investigation and discussion, and top students need to be encouraged to delve more deeply into the reasons and connections.
3. We need to encourage students, parents, teachers, and others in our society to believe that all students can learn mathematics and that our top students are capable of greater mathematical power than we have ever asked of them.

Curriculum

Opportunity to learn the curriculum is crucial to the development of gifted and talented students. It should have the following characteristics.

1. All students should follow the core curriculum recommended by the *Standards*. Top students should explore topics in more depth, draw more

generalizations, and create new problems and solutions related to each topic.

2. All students should have access to technology and manipulatives to aid in their construction of mathematical concepts. Top students should use these materials to explore even further.
3. Examples of superior student work should be available to students so they have something to strive for. Olympic athletes would not have progressed as far as they have, if they did not have superior examples of earlier athletes to emulate. The same is needed for student work in mathematics at all levels.

Teaching

Teachers are perhaps the single most important factor in the development of gifted and talented students. The following recommendations are made for teaching.

1. All teachers should follow the recommendations of the *Professional Standards* and encourage students to construct their own mathematical understanding, and teachers of the gifted and talented must encourage the highest levels of construction.
2. Teachers must learn to encourage and challenge their top mathematics students. They need adequate resources and support to obtain the materials, technology, and training they need to assist in the development of these students.
3. Students need a variety of rich, challenging mathematics programs from which to choose. They need to experience the joy of solving difficult mathematical problems and should be able to share that joy with others.
4. Parents and teachers should challenge students to ever-increasing levels of mathematical achievement. Teachers need to show students exemplary work from previous students so that students have examples of what can and should be accomplished.

Assessment

To assess mathematical talent and programs designed for gifted and talented students:

1. All teachers should follow the recommendations of the *Assessment Standards* and use a wide variety of assessment measures. The type of assessment used has a profound impact on the type of instruction offered and many standardized achievement tests limit the mathematics to low level computation. It is especially important for teachers of the gifted and talented to expect the highest levels of achievement on several types of assessment.

2. Students should use a wide range of technology and materials to produce quality mathematics. Today's technology allows students to create and display mathematics with outstanding merit.

Concluding Remarks

Our top students in mathematics are crucial to the well-being of our country. The only way we can meet our national goal of being first in the world in mathematics and science is to raise the mathematical competence of all our students, including the gifted and talented ones.

We must act immediately on a national level to upgrade the level of mathematics being offered to all our top students from kindergarten through graduate school. Perhaps, even more importantly, we must improve the ways in which our students learn mathematics. Teachers must become facilitators of learning to encourage all students to construct new, complex mathematical concepts. Students must be challenged to reach for ever-increasing levels of mathematical understanding. We must strive to help many more students including females, minorities, and students from rural and inner-city schools reach those top levels of mathematical ability. The potential exists in every school in our country for far more expertise in mathematics, and we must help students unlock their talents in this area.

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The Development of Gifted and Talented Mathematics Students and the National Council of Teachers of Mathematics Standards

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By the year 2000, U.S. students will be first in the world in science and mathematics achievement.
America 2000: An Education Strategy, 1991

In order to meet this lofty goal, we must make changes in our 110,000 public and private schools as well as in the homes and businesses of America. We must realize the importance of the development of gifted and talented students in our schools today and in the future who will be the leaders in the movement toward this goal. We are currently far from meeting this goal, and in order to reach it, changes in our attitudes, our curriculum, our teaching methods, and our means of assessment must be drastic and immediate. These changes must affect all students, including the gifted and talented.

Overview

In 1986, the Board of Directors of the National Council of Teachers of Mathematics established the Commission on Standards for School Mathematics as one means to begin these changes. This group developed a set of standards for the K-12 mathematics curricula in North American schools and a set of standards for evaluating the curricula and student achievement. The *Standards* were drafted in 1987, circulated among interested groups and revised in 1988, and published as the *Curriculum and Evaluation Standards for School Mathematics* in March, 1989. Currently, teachers, school district personnel, state level consultants, and many others across the United States and Canada are using these *Standards* as a basis for changes in their mathematics programs.

The *Curriculum and Evaluation Standards for School Mathematics* were followed by the *Professional Standards for Teaching Mathematics* in 1991. The *Professional Standards* acknowledged that teachers are the most powerful force in making changes in the ways in which mathematics is taught and learned in our schools and that they need adequate resources and long-term support in order to make these needed changes. The *Professional Standards* have numerous suggestions for ways that teachers can help students develop their mathematical talents.

The most recent of the *Standards* are the *Assessment Standards for School Mathematics*, with a working draft published in October, 1993. The *Assessment Standards* were designed to expand and complement, not replace, the Evaluation Standards that were part of the 1989 document. The *Assessment Standards* are based upon the assumption that every student is capable of achieving mathematical power (NCTM, 1993a, p. 3). This document includes a number of suggestions that can be applied to helping the top students develop even more power.

In this paper, ways to use all three sets of the *Standards* for the development of gifted and talented mathematics students are explored and discussed. Even though most areas of the three *Standards* do not directly address gifted and talented students, many of the recommendations are directly applicable. In 1993, NCTM released a draft of its position paper on *Provisions for Mathematically Talented and Gifted Students*. This paper was designed to replace an earlier position paper on gifted and talented and includes recommendations based on the *Standards*. The position paper stated:

> It is the position of NCTM that all students can benefit from an opportunity to study the core curriculum specified in the *Standards*. This can be accomplished by expanding and enriching the curriculum to meet the needs of each individual student, including the gifted (NCTM, 1993a).

Before looking at the recommendations from the *Standards*, research on the characteristics and identification of students displaying mathematical gifts and talents is reviewed. Students who show mathematical talent may or may not have similar gifts and talents in other areas. These students may be overlooked or underserved in the regular classroom or even in a general program for gifted students. At particular risk are females, minorities, and other traditionally underserved talented students in mathematics such as rural and inner-city students.

Because of the considerable number of recent reports on the poor showing of United States students on national and international mathematics tests, research on the level of mathematical competency of the gifted and talented students in the United States is also discussed. If the United States is going to remain a superpower in the world, we must pay more attention to our top mathematics students. We cannot be leaders in a technological world without the top schools and students in mathematical and technical fields.

An overview of the three sets of *Standards* is included along with recommendations for curriculum, teaching, and assessment of students showing mathematical talent. Specific recommendations for programs, teaching strategies, resources, and alternative means of evaluation suitable for gifted and talented mathematics students related to the *Standards* are suggested. It is hoped that teachers will take the suggestions and build upon them to meet the individual needs of their own students.

Gifted and Talented Mathematical Behaviors

Description of Behaviors

Mathematical talents are demonstrated in a variety of ways. Being a gifted or talented mathematics student has frequently been defined as scoring above the 95th percentile on a test of mathematical achievement. This is a very narrow definition of giftedness, however, and may contribute to the small number of successful mathematicians who are born and raised in the United States. In the United States, many students who perhaps do not score well on a standardized mathematics test decide at a very young age that they were not born with a mathematical mind, and therefore, they do not try to learn mathematics. Thus, mathematics achievement begins a downward spiral. This attitude has been socially acceptable and has contributed to the low esteem that many in the United States have for mathematical ability. Teachers, parents, and students must recognize the importance of mathematical ability for all students, and we must learn to recognize and develop these abilities in students of all races and socioeconomic groups, in girls as well as in boys.

Students who display mathematical talent often view the world from a mathematical viewpoint. Krutetskii called this a "mathematical cast of mind" (Krutetskii, 1976, p. 302). These students strive to make sense of the world by noticing spatial and quantitative relationships and connections in everything. This characteristic may show up as early as seven or eight years old. Krutetskii noted that these students may be analytic, geometric, or harmonic types (Krutetskii, 1976, p. 315-329). They may view the world abstractly using verbal-logical reasoning (analytic), or geometrically using visual expressions, or they may be able to use a combination of the two abilities (harmonic). Spatial abilities have not received the same emphasis as analytic abilities in mathematics until recently, and the NCTM *Standards* have recognized the need to develop both types of abilities in students.

General lists of characteristics of mathematical talents include characteristics of academic gifts in general such as fast learning pace, keen observation skills, powerful questioning skills, exceptional reasoning capacity, and creativity. Those characteristics specific to mathematics usually include the following:

- Early and keen awareness, curiosity, and understanding about quantitative information
- Ability to perceive, visualize, and generalize patterns and relationships
- Ability to reason analytically, deductively, and inductively
- Ability to reverse reasoning processes, and to switch methods easily but not impulsively
- Ability to work with mathematical concepts in fluent, flexible, and creative ways
- Energy and persistence in solving difficult problems
- Ability to transfer learning to novel situations
- Tendency to formulate mathematical questions not just to answer them

- Ability to organize and work with data in a variety of ways and to disregard irrelevant data (House, 1987; Greenes, 1981)

Notice that the list does not include the ability to compute rapidly and accurately. While some students showing mathematical talents may have this ability, it is not a necessary or sufficient characteristic of gifted mathematics students. Many gifted and talented mathematics students are impatient with details and do not care to spend time on computation. They are anxious to get on to the important aspects of the problems. Some may enjoy the challenge of being the fastest and most accurate on timed computation tests, but many are turned off by the low level of reasoning required.

We can think about mathematics students along a continuum or hierarchy as shown in the following diagram.

CREATORS
↑
PROBLEM POSERS
↑
PROBLEM SOLVERS
↑
CONSUMERS
↑
COMPUTERS
↑
DOERS
↑
ILLITERATES

(Sheffield, 1989)

At the bottom of the hierarchy are mathematical illiterates (or innumerates). Many people in the United States today seem almost to be proud of the fact that they do not understand or use mathematics. It is not uncommon to hear an adult say, "Oh, I always hated mathematics. I never was good at it." This attitude rubs off on students who assume it must be all right not to be able to do mathematics. This is a very dangerous belief, however, as our world becomes ever more dependent on a mathematically and technologically literate society.

Just above illiterates are students with the ability to do some computation with whole and rational numbers, called *doers*. These students have memorized rules for addition, subtraction, multiplication, and division and generally do fairly well on pages of computation in the math text as long as they simply have to repeat a process over and over again. They do not, however, generally understand why they are performing an operation in a certain way.

Above the level of doing mathematics, is the student who can compute well with all types of rational numbers and who understands the structure of the number system and the concepts of the operations. These students may receive high scores on the computation section of standardized tests of mathematical achievement. However, we cannot be satisfied with students who can just compute when any five dollar calculator would be faster and more accurate than most of the best human calculators.

Beyond the ability to compute is the ability to apply mathematical concepts to solve everyday problems. For a society to function, the people must be able to use mathematics every day both at work and at home. Students need to learn how to use mathematics in stores, in restaurants, painting a house, balancing a checkbook, and in a multitude of other situations. They must use mathematics to be intelligent consumers in today's society. It is these students who are sometimes identified as our top students by some standardized achievement tests that claim to include mathematical problem solving, but these tests do not identify the skills and processes necessary to be gifted and talented in mathematics.

It is above this level that solving real problems comes in. Problem solvers are able to apply their knowledge of mathematics in new situations where the answer is not obvious, and they have no preset rule to fall back on. They frequently use a method that they have not tried before or apply a method they have used to solve a completely different type of problem. In the past, we have asked our top students to work at this level, but today we realize that all students must be good problem solvers. Current standardized tests do not generally include this level.

An even greater talent than the ability to solve problems that someone else has suggested is the ability to create, define, or pose the problems. This talent relies on an ability to see important aspects of a situation and ask questions about it. Most of the mathematics known in the world today has been discovered in the last 50 years, and solutions to new problems would never have been found if someone had not suggested new problems upon which to work. We must encourage our top students to begin to work at this problem creation level. We cannot be satisfied with students merely solving problems that others have suggested. They must go on to suggest problems of their own. In acknowledging the importance of this level, we must include assessment and instruction that ask students to work at this level if we want to identify and develop high level mathematical talents. Students may be interested in beginning with the study of relatively new topics such as chaos and fractals or the new solution (apparently) to a problem posed over 100 years ago by Fermat.

At the top of the continuum are the creators of new mathematics. The creation of mathematics requires first the creation of new questions upon which to work and then the discovery or invention of the mathematics to answer the questions. Even young children can discover or create mathematics that is new to them, and they should be encouraged to do so. They will understand and remember the mathematics they have constructed for themselves much better than any of the mathematics we try to teach them. It is this level to which all our top students of any age should aspire. Not all students will be able to reach this level, but we should challenge our top students to strive for this whenever possible. It is only in this way that we will be able to move our society beyond its present level.

The ability to see relationships seems to be one of the main characteristics that separates expert problem solvers from novices. Students who are not good at solving problems tend to try to memorize rules and facts as unrelated bits of information. Good problem solvers look for the underlying structure and try to relate any new problem to information they already possess. This strategy should be modeled for students as they begin to learn to think mathematically.

In creating new problems, good problem solvers begin with information they understand and problems they have solved previously as a jumping off point for new questions. They view mathematics as a topic to be explored with rich new ideas waiting to be discovered or invented rather than a series of rules which they must memorize.

Good problem solvers may not be the most adept at computation. They take time to think about the problem before they begin to write anything. They do not give up easily if a problem is difficult. They view the problem as a challenge and enjoy working on it. If students have time to think about what they are doing and experience the joy of being successful at solving a difficult problem, all students can become better problem solvers, and the best need to be encouraged to be the creators of new mathematics.

Recommendations

In order to allow students to demonstrate that they are capable of top mathematical performance, we must do the following:

1. Give students a wide variety of rich, inviting tasks that require spatial as well as analytic abilities. Both of these are very important to success in mathematics. Good mathematicians must be skilled in both areas.
2. Encourage students to persist in solving mathematical problems. Difficult tasks require work for even the most gifted students. Many students give up before they give mathematics a chance.
3 Expect students to not only solve problems posed by others but to pose and solve new problems of their own. If students are only asked to compute, we will never know who can perform at the top levels in mathematics.

Identification of Mathematical Giftedness

Not all students with mathematical talent will have all the abilities listed above. Some students may exhibit some of these characteristics spontaneously, and others may display their talents only when presented with interesting problems. Many of the talents can be developed, and all students should be given problems where they can demonstrate their talents. In the past, many textbooks presented only low level computation exercises and simple one-step "word problems" that did not allow any students to display their talents with higher level thinking and creativity. The NCTM *Standards* recommend that teachers provide opportunities for all students to demonstrate and develop their mathematical talents. Teachers should be on the lookout for students that demonstrate talents in everyday problem solving.

Standardized Tests

Researchers at the National Center for Research on Mathematical Sciences Education (NCRMSE) began to examine mathematics assessment in the early 1980s and in comparing items to the NCTM *Standards* "demonstrated that the assessment procedures commonly used in schools were not only inadequate but should be viewed as a major barrier to the reform of school mathematics" (Romberg, 1993, p. 1). At a TERC-sponsored Exxon Conference on Alternative Assessment in K-3 Mathematics, one participant from the National Center for Fair and Open Testing stated that "testing damages children, and the harm that it causes far outweighs any possible benefits" (Mokros, 1991, p. 1). Most others at the conference agreed that there should be no standardized testing in math before fourth grade. They felt that such testing had a tendency to restrict the curriculum to computation and that for young children it was "not clear that standardized tests in fact tap mathematical skills; it is more likely that they tap children's verbal and writing skills instead" (Mokros, 1991, p. 18).

Standardized achievement tests may not identify students who are gifted and talented in mathematics. One reason is that these tests often concentrate on low level tasks. In a study by Romberg and Wilson (1992), six commonly used standardized tests for grade 8 (The Science Research Associates Survey of Basic Skills, The California Achievement Test, The Stanford Achievement Test, The Iowa Tests of Basic Skills, The Metropolitan Achievement Test, and The Comprehensive Test of Basic Skills) were examined to determine their alignment with the Curriculum Standards. Each item in the mathematics section of each test was classified for content, processes, and level based on the areas of the Curriculum Standards. From 62% - 82% of the items tested the content area of number with very few items relating to the other curricular areas of the Curriculum Standards. The process area of computation was tested by 62% - 91% of the questions, with again relatively few items testing other process areas. Procedures were tested by 84% - 96% of the items with only 4% - 16% of the items testing concepts. Therefore, students who test well on the mathematics area of these standardized achievement tests may or may not be students who would be classified as gifted and talented in mathematics based on the characteristics listed previously. These tests look at only a few of the content areas, processes, and concepts recommended by the NCTM

Standards. There are very few test items that test content areas such as probability and statistics or processes such as communicating and making connections. Areas that would allow students to display their mathematical talents are frequently missing from these standardized tests.

The most widely used tests for college entrance are the American College Testing program (ACT) and the Scholastic Assessment Tests (SAT). The ACT was revised in the late 1980s but continues to use multiple choice questions to test students' knowledge of mathematics. The Scholastic Assessment Tests will be used in their present form for the first time in Spring, 1994. Previously, the SAT was the Scholastic Aptitude Test and like the ACT used multiple choice questions to test student ability. The new SAT will allow students to use calculators on the mathematics portion of the test (NCTM, 1993b) and will include ten questions where students grid in answers. Even the new versions of these tests do not test many of the content areas recommended by the NCTM *Standards* and do not test processes such as problem solving and problem posing.

In spite of their limitations, these tests are perhaps the most widely used means of identifying mathematically talented students through the use of out-of-grade level testing. The Study of Mathematically Precocious Youth (SMPY), begun by Julian Stanley of Johns Hopkins University in 1971, identifies talented mathematics students through the use of the College Board's Scholastic Aptitude Test (SAT) or the American College Testing program (ACT). Through this talent search, students in seventh or eighth grade who test in the top 3% on a standardized achievement test are eligible to take the ACT or SAT to qualify as mathematically precocious. (Students may also be determined as verbally precocious through the use of these tests.) Because seventh and eighth graders who frequently have not had an algebra or geometry class are taking a test designed for above average high school juniors and seniors who generally have had such classes, Stanley claims that it can be assumed that this test is testing higher level thinking for these students. Stanley and Benbow (1986) assume that younger students taking this test must function more at an analytical reasoning level than high school juniors and seniors who may be functioning at Bloom's application level after completing one or more algebra or geometry classes. Seventh and eighth grade students taking this test have clustered in the top 3% on a grade level standardized achievement test but the SAT test scores spread these students out with a mean math score around 450 and a standard deviation near 100 (scores comparable to high school seniors). Barnett and Durden (1993, p. 167) reported after surveying 353 students who were identified as mathematically precocious through the use of SAT scores between 1980 and 1984, that these students subsequently maintained the potential identified by the test and actualized it through a myriad of varied intellectual experiences both in high school and in college. They cite this as validation and reinforcement of the use of the SAT for selection criteria.

Such out-of-grade level testing can identify a much wider range of mathematical talent than a grade level standardized test. Please note, however, that to qualify for the SMPY talent search, students must be in the top 3% of a traditional standardized test. Therefore, a number of mathematically talented students may never be identified because of the characteristics tested by the initial standardized test. With changes in both the SAT and the mathematics curriculum, the continuing effects of such testing remain to be seen.

A perhaps lesser known talent search is the U.S.A. Mathematical Talent Search directed by George Berzsenyi of Rose-Hulman Institute of Technology. Unlike the SMPY Talent Search, the U.S.A. Mathematical Talent Search sends out problems to students that they then have a month or longer to complete. The search includes several rounds of problems with each round containing five problems. Students are expected to send in solutions to at least two of the problems to then compete in the next round. This talent search is more in keeping with the NCTM *Standards* in that it not only fosters insight, ingenuity, and creativity, but it also rewards perseverance by allowing students to work on problems long enough to display their mathematical abilities. It is interesting to note that of the 865 students participating in the 1993-94 Talent Search, the most common last names of participants were (in descending order): Chen, Lee, Wang, Liu, Chang, Kim, Wong, Lin, Huang, Xu, Chan, Chiang, Jones, Park, and Wu. Rose-Hulman is in Indiana and the students are from the United States. Perhaps, there is a message here for the encouragement we are giving our best mathematics students (Berzsenyi, 1993).

Informal Identification

Other means of identifying mathematically talented students include observations, interviews, and self-, parent-, teacher-, and peer-identification. Some would argue that there is no need for formal identification of mathematically talented students if there are sufficient interesting problems, competitions, and other activities to stimulate and involve all students. A comparison to athletic abilities is often made. If students want to play basketball, there are sufficient areas where they can participate and develop their skills. All students can take part in informal basketball games outside of school and participate in intramural games during school. For students who display particular talents in this area, there are more formal competitions and teams where the students can sharpen their skills even further. A parallel in mathematics would be a society that encourages the mathematical participation of all its students in everyday activities outside of school and provides intriguing mathematical situations for all students in school. Students who show particular expertise in these activities are then provided with even more mathematical challenges in school and are encouraged to join academic teams and competitions for further enrichment of their mathematical talents. This model corresponds to Renzulli's Enrichment Triad (1977) that acknowledges the importance of creativity and task commitment as well as above average ability in mathematics. In this model, mathematical ability is not something that occurs at birth and is unchanging. It is assumed that all students can become more able mathematicians, and that using creative and higher level thinking and working hard on challenging problems are mathematical talents that can be developed and nurtured.

Arnold Ross, who has conducted programs for the mathematically talented for more than thirty years both at Notre Dame and Ohio State Universities, believes that mathematical talent is a learned behavior (Osborne, 1981). Several other directors of programs for top students in mathematics agree with him. The National Science Foundation funds Young Scholars Programs for middle and high school students who show promise of becoming mathematicians, scientists, and engineers. At a recent meeting of the directors of Young Scholars Programs in mathematics (Young Scholars, 1993), the characteristic distinguishing the best students in these mathematics programs across the country most frequently mentioned by the directors was persistence. The directors all agreed that students who were willing to stick with a problem and not give up were the top students in every program. It should be noted that students were chosen for the programs partially based on previous demonstrated abilities to do above average work in mathematics. Once chosen, however, the students with the top scores on standardized mathematics tests were not always the top students in Young Scholars Programs. Persistence and hard work were more indicative of top performance.

Many of the Young Scholars Programs also allowed for students to work cooperatively with other bright, hard-working students as well as with university scholars and researchers. This ability to work well with others and to learn from each other also proved to be important to the production of quality mathematical products and research.

If we do not provide students with the opportunity to hone this talent through practice with stimulating problems, we may well be missing a tremendous opportunity for the development of mathematical abilities in the United States. We must stimulate all students in mathematics and expect our best students to work at a level far above that currently expected.

Recommendations

To identify mathematical talent, the following steps are recommended:

1. We must use a variety of identification measures. Standardized tests measure only a very narrow range of generally low level skills.
2. We must provide students with assessment tasks that tap skills beyond computation. These tasks can frequently not be measured by paper and pencil multiple choice tests.
3. We must have a wide range of opportunities for more informal identification of mathematical talent such as exciting mathematics classes, mathematical clubs, and contests where students can demonstrate and hone their mathematical abilities.

Status of Mathematical Talent in the United States

National Comparisons

A great deal has been written recently about the status of students in the United States in the areas of mathematics and science. Much of the research focuses on all students, however, and does not consider the plight of our top students. In some areas, teachers, parents, and school administrators still believe that the gifted students can and do perform satisfactorily on their own. Research shows that this is not the case, however.

Note that much of the research reported in this section involves the use of standardized tests that were discussed in the previous section. It is acknowledged that the use of these tests in reporting on the success of our top students is problematic; however, there are very few studies that use other measures.

Singal (1991) claimed that beginning in the mid-1970s students have been entering college so poorly prepared that some are almost dysfunctional in mathematics. He claims that our brightest youngsters have suffered the most dramatic setbacks over the past two decades and that this severely affects our ability to compete with other nations in the future.

In spite of a dramatic drop in top scores on the verbal portion of the Scholastic Aptitude Test (SAT), the percent of top scores on the quantitative portion of the test has risen steadily (Turnbull, 1985). The reasons for this encouraging trend are difficult to ascertain. The average number of schools requiring at least three years of high school math have increased in the last 20 years and this should have a positive effect on SAT scores. The larger number of Asian students taking the exam has probably also had an effect on the number of high scores. In 1972, 1% of the students taking the SAT were Asian. By 1990, this number had risen to 7%. The average quantitative score of Asian-American students in 1990 was 528, 37 points above that of the average white student score (Callahan, 1992). Another possible explanation for the increase in high scores on the quantitative section of the SAT is the increase in students identified as mathematically talented through the administration of the SAT (and more recently the ACT, NCTM, 1993c) to seventh and eighth grade students through the Talent Search begun by Julian Stanley. Through this program, more students have been able to take advantage of enrichment programs and accelerated or advanced courses in mathematics.

The National Assessment of Educational Progress (NAEP) also showed that math scores rose between 1982 and 1988 (Educational Testing Service, 1990), although most increases were on lower level skills. Unlike other standardized tests, the National Assessment of Educational Progress contains items testing complex mathematical procedures and reasoning as well as basic computation on simple problem solving. This is still a paper and pencil, multiple choice test, however. The test showed 98% of 17-year-olds had mastered basic operations and beginning problem solving (Level 250) in 1988 as compared to 93% in 1982. Fifty-nine percent had mastered moderately complex procedures and reasoning (Level 300) in 1988, and only 48% had mastered similar skills in 1982. However, only 7% of the 17-year-olds were proficient in the area of multistep problem solving, geometry, and basic algebra (Level 350), an insignificant increase since 1982. In addition, virtually no 13-year-olds were proficient at this level. The fact that virtually none of our 13-year-olds and only 7% of our 17-year-olds are proficient in multistep problem solving, geometry, and algebra indicates that "only a small fraction of high school students leave high school prepared to enter quantitative fields in college, or to do the kind of statistical quality control work increasingly required in factories"

(Educational Testing Service, 1990, p. 14). Even our very best students are woefully underprepared for any type of quantitative work.

Unfortunately, even though mathematics scores are rising, fewer students are choosing careers in math, science, and engineering. In 1982, half of the students scoring in the top 10% on the SAT planned to major in math, science, or engineering. Only 44% had such plans in 1986. For females, this number is even more discouraging. In 1986, only about 15% of the white females in the top 10% on the SAT planned to major in a highly quantitative field (Grandy, 1987).

At the graduate level, the decline of American-born students pursuing advanced degrees in mathematics is dramatic. Even though the total number of students enrolled in graduate programs in mathematics in the United States increased from 1975 to 1986, the number of U.S. students declined by 1400 while the number of non-U.S. students increased by 3100 (Madison & Hart, 1989). The percent of doctorates earned by U.S. citizens declined from 72.3% of the total to 50.3% of the total while the number of doctorates earned declined from 1211 to 730 (National Research Council, 1987).

International Comparisons

As reported by Callahan (1992), Miwa noted that the type of items from Level 350 of the National Assessment of Educational Progress test are asked of fifth and sixth grade students in Japan. More than 60% of Japanese students under the age of 13 can answer these questions, while only .4% of American 13-year-olds and 7% of American 17-year-olds can answer these questions. The level of work that is expected of fifth and sixth grade Japanese students is not introduced in most American schools until high school and is mastered by only 7% of American high school graduates.

The fact that math scores in the United States have gone up in the past 20 years means little when we compare our test results to those in other countries. Typical math questions on the ACT and SAT require far less mastery of mathematics than similar tests in other countries (Wu, 1993). Teachers and parents do not expect nearly as much of U.S. students as do teachers and parents in the rest of the world.

Perhaps the most disturbing indicator of how our top mathematics students are doing comes from international studies. The Second International Mathematics Study (Crosswhite, et al., 1986) showed that performance of the top 5% of U.S. students in college preparatory mathematics is exceeded by 50% of the students in Japan. The International Association for the Evaluation of Educational Achievement (IEA) (McKnight, et al., 1987) showed the top 3% of American students only earned scores at the average of all students taking the same level of mathematics in other countries. Our very best students—the top 1%—scored lowest in algebra of the top 1% in all participating countries. Of the top 5%, U.S. students were above only the top 5% of students in Israel. In functions and calculus, the top 1% of U.S. students scored only a few points above students from British Columbia, Canada, even though calculus is not part of the Canadian curriculum.

Robitaille and Travers (1992, p. 707) point to the importance of the longitudinal study that was included in the Second International Mathematics Study. The longitudinal study showed that for topics that were emphasized in grade eight in France and Japan, the students had extremely high-growth rates in test scores. Similarly, they point to the consistently high levels of achievement attained by students from some countries that they note should be a spur and incentive for other countries to improve. They underscore the importance of the opportunity to learn; if topics and processes are not taught, students do not learn them.

Poor showings on international comparisons are not limited to high school students. Stevenson, Lee, and Stigler (1986) compared mathematics achievement of Japanese, Chinese, and American first and fifth graders. In first grade, only 15 Americans were among the top 100 scorers and by fifth grade only one American was among the top 100 scorers. Americans were found to be as inefficient at computation as they were at solving word problems. Stevenson, et al., found that the average amount of time in American first and fifth grade classrooms that was spent on mathematics was about three hours a week, less than half the amount of time spent on math in Japan and China. They also found the type of instruction to differ between the two countries. Verbal explanations on the part of both the students and the teachers was much greater in Japan than in the United States. In Japan, the most common practice when students made an error was to ask the student to put the problem on the board and then to discuss with the entire class the processes that led to the error. In the U.S., the most common way of evaluating student work in the classroom was to praise students with correct answers and to ignore errors. In addition, the only classes where entire class periods of forty minutes or more were spent on one or two problems were in Japan; Japanese teachers did not rush through material but regularly stopped to discuss and explain it.

American mothers were generally pleased with the job that the schools were doing in mathematics, unlike the Chinese and Japanese mothers who were not as pleased with the schools. In addition, American mothers generally attributed their children's success or lack of it in mathematics to the ability of the child, whereas in Japan and China, success was attributed to effort. This is related to statements from the directors of Young Scholars mathematics programs that effort and persistence were the most significant indicators of mathematical talent. We are selling students short in the United States by allowing them to believe that mathematical talent is something that you are born with that cannot be developed.

From these and other studies, it appears that as early as first grade, U.S. students (including the very best students) perform more poorly than Asian students and this trend continues through high school and beyond. This is apparently due to a combination of factors that include low expectations on the part of parents, teachers, and the students themselves, a belief that success in mathematics is due to ability rather than effort, a repetitive curriculum that does not expose children to new, more difficult concepts, a lack of time to reflect upon and discuss mathematical problems, and a lack of classroom instruction and homework time devoted to mathematics.

Recommendations

In light of national and international studies of mathematics achievement, the following recommendations can be made:

1. Students in the United States need the opportunity to learn more mathematics. This is especially true of our top students who need the challenge of new and more complex problems, rather than the repetition of a typical mathematics curriculum.
2. Teachers and students in the United States need to tackle fewer problems, but in far greater depth. Time is needed for investigation and discussion, and top students need to be encouraged to delve more deeply into the reasons and connections.
3. We need to encourage students, parents, teachers, and others in our society to believe that all students can learn mathematics and that our top students are capable of greater mathematical power than we have ever asked of them.

National Council of Teachers of Mathematics

Position Paper on *Provisions for Mathematically Talented and Gifted Students*

In addition to the NCTM *Standards*, the National Council of Teachers of Mathematics has an Instructional Issues Committee that considers current issues and writes position papers on the NCTM stand on these issues. These are studied and discussed by the mathematics education community before being officially adopted by NCTM. In 1993, the Instructional Issues Committee drafted a new position paper on *Provisions for Mathematically Talented and Gifted Students*. The draft of this paper is included in Appendix A. This paper acknowledges the importance of opportunities for gifted and talented students to develop to their full potential. *Provisions for Mathematically Talented and Gifted Students* (NCTM, 1993d) states that "while all students need curricula that develop the students' problem solving, reasoning, and communication abilities, the mathematically talented and gifted need in-depth and expanded curricula that emphasize higher order thinking skills, nontraditional topics, and the application of skills and concepts in a variety of contexts. . . . Therefore, the National Council of Teachers of Mathematics recommends that all mathematically talented and gifted students have access to appropriate curricula and instruction that contributes to developing positive attitudes, furthering their mathematical interests, and encouraging their continuing participation in the study of mathematics." Appropriate curricular suggestions, instructional strategies and assessment ideas will be considered in the following sections.

Curriculum and Evaluation Standards for School Mathematics

Overview

In 1986, the National Council of Teachers of Mathematics (1989a) appointed a Commission on Standards for School Mathematics and charged it with two tasks:

- Create a coherent vision of what it means to be mathematically literate in a world that relies on calculators and computers to carry out mathematical procedures, and in a world where mathematics is rapidly growing and is extensively being applied in diverse fields.
- Create a set of standards to guide revision of the school mathematics curriculum and associated evaluation toward this vision. (p. 5)

These *Standards* were developed by the committee and disseminated to the mathematics education community in the United States and Canada. After much discussion and many revisions, the final *Standards* were published in 1989. The *Standards* have been the guidelines for the revision of mathematics education across the United States. They have been used to guide revisions in the curriculum, the instruction, and the means of assessment.

The *Standards* include separate recommendations for grades K-4, 5-8, and 9-12. All recommendations are based upon the following assumptions:

1. Mathematical power can and must be at the command of all students in a technological society.
2. Mathematics is something one does—solves problems, communicates, reasons; it is not a spectator sport.
3. The learning of mathematics is an active process, with students constructing knowledge derived from meaningful experiences and real problems.
4. A curriculum for all includes a broad range of content, a variety of contexts, and deliberate connections.
5. Evaluation is a means of improving instruction and the whole mathematics program.

Five goals are included for students of all ages:

1. to learn to value mathematics,
2. to learn to reason mathematically,
3. to learn to communicate mathematically,
4. to become confident of their mathematical abilities, and
5. to become mathematical problem solvers.

Due to the advances in technology, the following recommendations are made for students of all ages:

1. Appropriate calculators should be available to all students at all times.
2. A computer should be available in every classroom for demonstration purposes.
3. Every student should have access to a computer for individual and group work.
4. All students should learn to use the computer as a tool for processing information and performing calculations to investigate and solve problems.

Acknowledging a constructive, active view of the learning process, it is recommended that instruction should vary and include opportunities for:

1. appropriate project work,
2. both group and individual assignments,
3. discussion between teacher and students and among students,
4. use of concrete, manipulative materials. (NCTM, 1989b)

In the area of assessment, the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989b, p. 190) propose that:

1. student assessment be integral to instruction,
2. multiple means of assessment methods be used,
3. all aspects of mathematical knowledge and its connections be assessed,
4. instruction and curriculum be considered equally in judging the quality of a program.

The Evaluation Standards recommend that some types of evaluation be emphasized more while others are de-emphasized. These are summarized in the following chart.

Emphasis of the Evaluation Standards

Increased Attention	**Decreased Attention**
• Assessing what students know and how they think about mathematics.	• Assessing what students do not know.
• Having assessment be an integral part of teaching.	• Having assessment be simply counting correct answers on tests for the sole purpose of assigning grades.
• Focusing on a broad range of mathematical tasks and taking a holistic view of mathematics.	• Focusing on a large number of specific and isolated skills organized by a content-behavior matrix.

Emphasis of the Evaluation Standards (*continued*)

Increased Attention	**Decreased Attention**
• Developing problem situations that require the applications of a number of mathematical ideas.	• Using exercises or word problems requiring only one or two skills.
• Using multiple assessment techniques, including written, oral, and demonstration formats.	• Using only written tests.
• Using calculators, computers, and manipulatives in assessment	• Excluding calculators, computers, and manipulatives from the assessment process.
• Evaluating the program by systematically collecting information on outcomes, curriculum, and instruction.	• Evaluating the program only on the basis of test scores.
• Using standardized achievement tests as only one of many indicators of program outcomes.	• Using standardized achievement tests as the only indicator of program outcomes. (NCTM, 1989b, p. 191)

The separate *Standards* curriculum recommendations for grades K-4, 5-8, and 9-12 include the goals for all students listed above and the following:

NCTM *Curriculum Standards* for Grades K-4

1. Mathematics as Problem Solving
2. Mathematics as Communication
3. Mathematics as Reasoning
4. Mathematical Connections
5. Estimation
6. Number Sense and Numeration
7. Concepts of Whole Number Operations
8. Whole Number Computation
9. Geometry and Spatial Sense
10. Measurement
11. Statistics and Probability
12. Fractions and Decimals
13. Patterns and Relationships

K-4 Assumptions

1. The K-4 curriculum should be conceptually oriented.
2. The K-4 curriculum should actively involve children in doing mathematics.
3. The K-4 curriculum should emphasize the development of children's mathematical thinking and reasoning abilities.
4. The K-4 curriculum should emphasize the application of mathematics.
5. The K-4 curriculum should include a broad range of content.
6. The K-4 curriculum should make appropriate and ongoing use of calculators and computers. (NCTM, 1989b, pp. 17-19)

NCTM *Curriculum Standards* for Grades 5-8

1. Mathematics as Problem Solving
2. Mathematics as Communication
3. Mathematics as Reasoning
4. Mathematical Connections
5. Number and Number Relationships
6. Number Systems and Number Theory
7. Computation and Estimation
8. Patterns and Functions
9. Algebra
10. Statistics
11. Probability
12. Geometry
13. Measurement

5-8 Assumptions

1. Every classroom will be equipped with ample sets of manipulative materials and supplies.
2. Teachers and students will have access to appropriate resource materials from which to develop problems and ideas for explorations.
3. All students will have a calculator.
4. Every classroom will have at least one computer available at all times for demonstrations and student use. Additional computers should be available for individual, small group, and whole class use. (NCTM, 1989b, pp. 67-68)

NCTM *Curriculum Standards* for Grades 9-12

1. Mathematics as Problem Solving
2. Mathematics as Communication
3. Mathematics as Reasoning
4. Mathematical Connections
5. Algebra

6. Functions
7. Geometry from a Synthetic Perspective
8. Geometry from an Algebraic Perspective
9. Trigonometry
10. Statistics
11. Probability
12. Discrete Mathematics
13. Conceptual Underpinnings of Calculus
14. Mathematical Structure

<u>9-12 Assumptions</u>

1. Students entering grade 9 will have experienced mathematics in the context of the broad, rich curriculum outlined in the K-8 standards.
2. The level of computational proficiency suggested in the K-8 standards will be expected of all students; however, no student will be denied access to the study of mathematics in grades 9-12 because of a lack of computational facility.
3. Although arithmetic computation will not be a direct object of study in grades 9-12, number and operation sense, estimation skills, and the ability to judge the reasonableness of results will be strengthened in the context of applications and problem solving, including those situations dealing with issues of scientific computation.
4. Scientific calculators with graphing capabilities will be available to all students at all times.
5. A computer will be available at all times in every classroom for demonstration purposes, and all students will have access to computers for individual and group work.
6. At least three years of mathematical study will be required of all secondary school students.
7. These three years of mathematical study will revolve around a core curriculum differentiated by the depth and breadth of the treatment of topics and by the nature of applications.
8. Four years of mathematical study will be required of all college-intending students.
9. These four years of mathematical study will revolve round a broadened curriculum that includes extensions of the core topics and for which calculus is no longer viewed as *the* capstone experience.
10. All students will study appropriate mathematics during their senior year. (NCTM, 1989b, pp. 124-125)

These are curricular areas recommended for study by all students. In the past, much of the curriculum for gifted students included areas of study that were frequently not included in the curriculum for all students such as probability and statistics. These areas are now recommended for everyone, not just the gifted. The recommended methods of teaching for all students also include techniques that were at one time restricted to gifted students such as asking students to reason and explain their thinking, using concrete models to demonstrate mathematical concepts, and making and testing hypotheses about the nature of mathematics. With these techniques now recommended for everyone, what should be done for gifted and talented students?

Standards Recommendations for the Gifted and Talented

The *Standards* do acknowledge the gifted. The *Standards* (1989b) state:

> This, however, does not suggest that we believe all students are alike. We recognize that students exhibit different talents, abilities, achievements, needs, and interests in relationship to mathematics. The mathematical content outlined in the *Standards* is what we believe all students will need if they are to be productive citizens in the twenty-first century. If all students do not have the opportunity to learn this mathematics, we face the danger of creating an intellectual elite and a polarized society. (p. 9)

> On the one hand, prior to grade 9, we have refrained from specifying alternative instructional patterns that would be consistent with our vision. On the other hand, for grades 9-12, the *Standards* have been prepared in light of a core program for all students, with explicit differentiation in terms of depth and breadth of treatment and the nature of applications for college-bound students . . . their experiences may differ in the vocabulary or notations used, the complexity of arguments, and so forth.

The *Standards* contain a warning for advanced high school students:

> A school curriculum in line with these standards should be organized so as to permit all students to progress as far into the mathematics proposed here as their achievement with the topic allows. In particular, students with exceptional mathematical talent who advance through the material more quickly than others may continue to college level work in the mathematical sciences. However, we strongly recommend against acceleration that either omits content identified in these standards or advances through it superficially. (p. 124)

Even though the *Standards* recommend high standards for all students, that does not mean that all students are expected to reach the same levels in the same way. All students should study the same curricular topics, but some students should be expected to study the topics in greater depth, making more connections and generalizations than others. Students may want to study some topics in more depth than other topics. Just because a student displays mathematical talents does not mean that that student will want to or should be expected to master every topic studied to the same degree. The student should be expected to ask nonsuperficial questions about the topic and to explore a variety of means of answering those questions, however, for all strands of the *Standards*.

Curriculum Suggestions

Talented students should always be encouraged to think more deeply about mathematical topics being studied. As students create projects and investigations, teachers should keep copies of the very best work each year. When students see examples of outstanding work, they realize what is expected of them. They then endeavor to improve upon others' best efforts. Each year, the best work should get even better.

Rich topics should supply a never ending source of questions and relationships for students to think about. The following list of topics might give students a jumping off point for their research. Sources of more information on these and other topics are included in the reference list and in Appendices B and C. These are only a beginning, and teachers and students should develop other ideas of their own. Some of these are traditional topics that mathematicians have studied for centuries and others are relatively new, but all offer a wealth of possible connections and related areas of study:

Fractals and Chaos
The Pythagorean Theorem and Pythagorean Triples
Fibonacci Numbers
Finite Differences
Pascal's Triangle
Golden Rectangle
Magic Squares
Other Numeration Systems (Babylonian, Egyptian, Chinese, Mayan, Roman, etc.)
Figurate Numbers (triangular, square, pentagonal numbers, etc.)

Lives and Research of Famous Mathematicians such as: Karl Gauss, Benjamin Banneker, Ada Lovelace, Leonard Euler, Albert Einstein, R. Buckminster Fuller, Moritz Cantor, Jules Poincare, Charles Babbage, Adrien Marie Legendre, George Boole, Johann Kepler, Sir Isaac Newton, Jakob Bernoulli, and Sonya Kovalevsky
Pi
Platonic Solids
Topology
Pendulums and Other Applied Physics Problems
Transformational Geometry
Proportional Reasoning—levers, means, mixtures, batting averages, speeds, similar figures, scale drawings, etc.
Design Problems
Combinatorics
Graph Theory
Aristotelian Logic and Matrix Logic Problems
Relationships among Constructions on a Quadrilateral or Triangle such as Medians, Perpendicular Bisectors, Angle Bisectors, etc.
Computer Programming and Robotics including Logo and LEGO-Logo

Any projects chosen should be related to the planned curriculum that should be taken from one of the strands of the NCTM *Standards*. Teachers should keep a file of interesting problems that they have used with students or that they find interesting to study on their own. Students and teachers alike should regularly read journals and other publications such as *Teaching Children Mathematics* (elementary), *Mathematics Teaching in the Middle School*, the *Mathematics Teacher* (secondary), and *Student Math Notes* from NCTM, *Quantum* (secondary), The HiMap Pull-out Section of Consortium (high school) and *The Elementary Mathematician* from Co-Map, *Hands On!* (all levels) from TERC, MathCounts booklets (middle school), *Math Horizons* for college undergraduates from the Math Association of America, and the *School Science and Mathematics* journal (elementary and secondary) to get new ideas for further investigation.

The Core Curriculum

NCTM has published a series of Addenda books that are a good source of ideas for topics related to each area of the Curriculum Standards for students of all ages. Many of these are very rich, challenging topics for gifted students. The Addenda books that are perhaps the most applicable for high school teachers of talented students are those that address the Core Curriculum.

The NCTM Curriculum Standards recommend that all high school students study the same core curriculum in the first three years of high school. Some expanded topics are added for college bound students in the senior year. This Core Curriculum recommends that:

> . . . differentiation in learning outcomes occurs by blending core lessons for all students with extended activities that students can complete to different depths and levels of abstraction and formalism. As should be the case with all student investigations, provisions are made for students to share their experiences, clarify their thinking, generalize their discoveries, and construct convincing arguments. (Hirsch, 1992, p. vii)

In a core curriculum, therefore, all students study a rich, nonrepetitive curriculum, with the top students being challenged by extended activities that require greater depth of exploration and generalization.

Criteria for Challenging Mathematics Curricula

In judging whether curriculum for gifted and talented students in mathematics is appropriate, it should meet the following criteria. Many of the criteria are recommended for all students, and all students should be encouraged to explore mathematical topics as deeply and thoroughly as possible.

1. The curriculum should challenge students to use and explain logical, inductive and deductive reasoning.

2. Curriculum materials should encourage students to ask questions and make generalizations that go beyond the original problem.
3. Curriculum should suggest a variety of methods, materials and technology to solve a given problem. Even the very best students should not be restricted to paper and pencil or mental problem solutions.
4. The curriculum should cover all the areas of the *Standards* recommended for that grade level. Topics such as geometry, algebra, statistics, and probability should be integrated, not studied in isolation. Many topics should be explored in greater depth by top students.
5. Measurement of achievement should include a wide range of measures such as observations, interviews, exhibitions, demonstrations, portfolios, open-ended questions, and performance events. These will be discussed in more detail in the section on assessment.
6. There should be opportunities for students to work with others at their achievement level as well as opportunities for students to do independent research and investigations. Bright students need the opportunity to discuss mathematical concepts with others of their developmental level.
7. Assignments should be flexible enough to allow students to demonstrate mastery of topics such as low level computation skills and allow for more extensive, deeper investigation of interesting topics.
8. Lectures and repetition should be avoided or kept to a minimum to allow students time and opportunities to be coinvestigators with the instructor and with each other.
9. Extensive use should be made of concrete, manipulative materials for students of all ages. Students should make abstract generalizations based on their concrete manipulation of materials when possible.
10. Expectations should be very high. Students should be expected to go beyond previous levels with each new task. They should make connections to previous mathematics, other subjects, and everyday life, and use elegant, precise, accurate language to describe their thinking and their results. They should be challenged to extend and generalize new learning whenever possible.

The list of publishers in Appendix B have materials that meet some, if not all, of these criteria. In addition, the references list several other sources of information for helping gifted students develop mathematics abilities.

Recommendations

Opportunity to learn the curriculum is crucial to the development of gifted and talented students. It should have the following characteristics.

1. All students should follow the core curriculum recommended by the *Standards*. Top students should explore topics in more depth, draw more generalizations, and create new problems and solutions related to each topic.

2. All students should have access to technology and manipulatives to aid in their construction of mathematical concepts. Top students should use these materials to explore even further.
3. Examples of superior student work should be available to students so they have something to strive for. Olympic athletes would not have progressed as far as they have, if they did not have superior examples of earlier athletes to emulate. The same is needed for student work in mathematics at all levels.

Professional Standards for Teaching Mathematics

Overview

In 1991, the *Professional Standards for Teaching Mathematics* was published as a companion document to the *Curriculum and Evaluation Standards for School Mathematics*. This document gives insight into the direction NCTM has recommended for mathematics instruction. As with the *Curriculum Standards*, the *Professional Standards* contain recommendations for teaching all students, but many of the suggestions are invaluable to teachers of our best students. The *Professional Standards* are based upon the following assumptions:

- Teachers are key figures in changing the ways in which mathematics is taught and learned in schools.
- Such changes require that teachers have long-term support and adequate resources.
- Effective teachers are those who can stimulate students to learn mathematics.
- Students learn mathematics well only when they construct their own mathematical understanding. (NCTM, 1991, p. 2)

The *Professional Standards* recommend five major shifts in the environment of the mathematics classrooms to allow for the empowerment of students. They recommend a shift:

- toward classrooms as mathematical communities—away from classrooms as simply a collection of individuals;
- toward logic and mathematical evidence as verification—away from the teacher as the sole authority for right answers;
- toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
- toward connecting mathematics, its ideas and its applications—away from treating mathematics as a body of isolated concepts and procedures. (NCTM, 1991, p. 3)

Teachers of the gifted will recognize many of these recommended shifts as long-time practices in gifted and talented programs. These shifts are now being recommended for teachers of all students. It is noted in the *Standards*, "however, this does not mean that every child will have the same interests or capabilities in mathematics" (NCTM, 1991, p. 4). Some students will continue to perform at higher levels than others, and we must raise our expectations for these top students as well.

Teaching Heuristics and Strategies

Because the *Standards* are directed at teachers of all students, it is useful for teachers to have teaching heuristics and strategies designed specifically for the development of mathematical talent. A heuristic is a general method of solving a problem. Heuristics can help students in getting started on the solution to a problem when they might otherwise give up because they did not know where to begin.

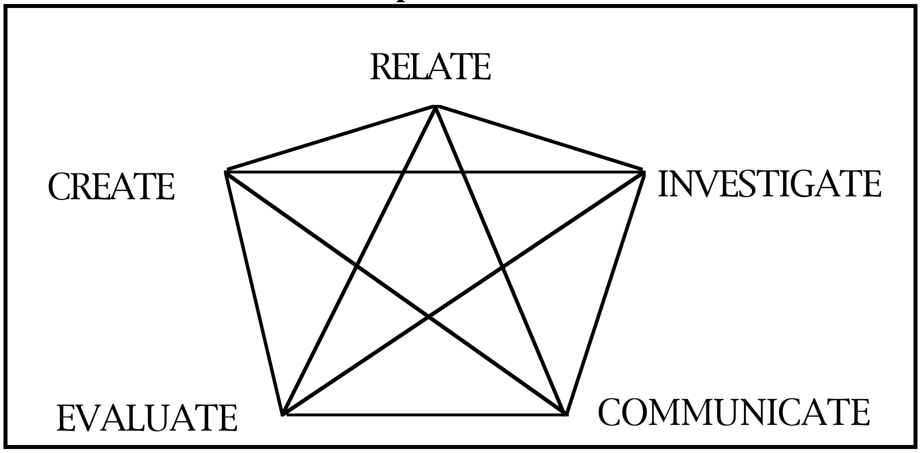
Perhaps the most famous heuristic for teaching problem solving was developed by Polya (1957). He outlined four steps in the problem solving process:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

The first step is obvious, yet students are often frustrated because they do not understand what the problem asks. The second step encourages the students to reflect upon the problem and plan their next steps. In the third step, this plan is put into action, and at the end, the problem is reviewed, the answer is studied to make sure it is reasonable and that loose ends are tied up. This process does not necessarily happen in order. Students frequently begin to carry out the plan and realize that they have reached a dead end. At this point, they must go back and devise another plan or perhaps return to the original problem to see if they understood it correctly.

For encouraging top students, a heuristic that goes beyond Polya's final step of looking back is needed. One heuristic that has proven to be useful for both the creation and the solution of problems is the following Model for the Development of Mathematical Talent (Jensen, 1976). Notice that this is not a linear model. Students can move from any point on the figure to any other point as the need arises in solving the problem.

Model for the Development of Mathematical Talent



Students may begin at the RELATE step. In this step, a student uses all available information that relates to the mathematical area on which he or she is working. For example, a student studying prime numbers may study the sieve of Erastothenes, greatest common divisors or factors, least common multiples or denominators, composite numbers, even and odd numbers, and other number theory topics. After a student has investigated a number of related areas, he or she may create a new question on which to work. For instance, a student might wonder whether the sieve of Erastothenes would work in other number bases. The student would then investigate this process in other bases, perhaps base five and base twelve, looking for similarities and differences among the different bases with possible returns to look at other relationships or to create other questions to study. After a thorough investigation, the solution or solutions are evaluated and promising solutions are reported to any interested individuals. This may include classmates, younger students, professional mathematicians, or mathematics educators (through journals or conferences), interested parties in industry, and teachers. Teachers should help supply talented students with outlets for sharing the results of their research. The National Council of Teachers of Mathematics and the Mathematics Association of America will consider students' work for publication in their journals. In addition, the journal *Quantum*, a joint venture between the United States and Russia, publishes mathematics articles by and for high school students. Many schools publish their own monthly, quarterly, or annual newsletter or journal with the latest results from the students' research. Students may then use the results from their own or other students' investigation to stimulate the creation of other questions to study.

Program Options for Gifted and Talented Mathematics Students

There are a variety of ways to provide for the needs of gifted and talented students in mathematics. A good program should include several of the following to allow students and parents to choose the means most suited to the students' individual needs.

In-Class Programs

Whenever students are heterogeneously grouped in mathematics classes, provisions need to be made for students exhibiting gifted and talented behaviors. The NCTM *Standards* provide guidelines for a much richer program for all students, but some students should be expected to go even further than others. High expectations should be held for all students, but that does not mean the same expectations for all. All students should be challenged to work at their highest level, and this means that as the expectation level rises for the average students in the class, it must rise even higher for the top students. Problems given to the class should be rich enough to allow for solutions on a variety of levels. Questions should be open-ended and allow for individual exploration and investigation. Suggestions from the Core Curriculum for differentiated assignments and investigations are useful for challenging top students.

In heterogeneous classrooms, time is needed for talented students to work together on problems. This might be accomplished through the use of cluster grouping within the class that allows gifted students to work together to investigate challenging problems. Time might be made for in-depth investigations through curriculum compacting of low level computation skills and flexible assignments that allow students to demonstrate mastery of topics that the rest of the class might be studying while the top students work together on new, more challenging tasks.

Technology should be readily available to assist students in their investigations. Technology should include, at the minimum, computers with modems and challenging software, and appropriate calculators. By middle school, students should have graphics calculators available for their explorations. Other technology might include interactive laser or video discs and CD-Roms. Students should have access via modems to students and data bases around the world to use in their research.

Pull-out Programs

Pull-out programs are frequently used in elementary school where students spend most of the time in regular classrooms and part of the time in classes with other gifted and talented students. These programs can be effective in mathematics if students are selected for the programs based on mathematical talent and the programs are designed to build on those talents. Mathematics curriculum in these classes need to be coordinated with the mathematics curriculum in the regular classroom whether the pull-out program is designed to replace or to supplement the regular program.

Magnet Schools

Magnet schools frequently involve students from a region or whole state and focus on one or more areas such as magnet schools for math and science. At the elementary and middle grades levels, these are generally regional schools where all the students commute; but at the high school level, these may be local schools that students attend only for mathematics and science while other classes are taken at a home school such as several of the regional Virginia Governor's Schools, regional schools such as the Bronx High School of Science or Summatech in Minneapolis where students commute and take all their classes in one school, or statewide schools such as the North Carolina

School of Science and Mathematics that is a residential school for the top eleventh and twelfth grade students from the entire state. These schools frequently hire top teachers with special training in math or science as well as training in working with talented and gifted students. Expectations are very high and competition among the students helps produce high level products.

Acceleration

Students may be accelerated in their mathematics program at any stage in their school career. This may involve early entrance to any level of school: kindergarten, middle school, high school, or college; grade skipping, or simply doing math with students in a higher grade. Extreme care must be taken with acceleration, however. If the program at the higher grade level is simply doing more low level math such as accelerating a six-year-old into a third grade class that is memorizing multiplication facts, or accelerating a middle school student into a geometry class that is memorizing proofs, this is not appropriate. The higher grade level class should be doing math that requires higher level thinking that is more on the level of the student. Classes such as this are relatively rare; frequently the younger student is thinking on a higher level than the older student. The mathematics older students are doing may have a few more prerequisite concepts, but the thinking required is not higher level.

With high school students, care must be taken that the students are not accelerated through math classes so fast that the students spend a year or more in high school with no math classes left to take. If all high school math classes have already been mastered by the student, the school should assist the student in finding other appropriate math classes. This may involve enrolling in math classes at the local university, working through independent study, taking correspondence courses or finding a distance learning class that may be offered through another school or university via cable television. The NCTM *Standards* recommend more of a Core Curriculum approach where all students study a basic core of mathematics, but assignments are differentiated to provide for higher level, more challenging problems and investigations for the top students.

At the high school level, two of the most popular ways to meet the needs of top students are Advanced Placement (AP) courses and the International Baccalaureate (IB) program. The AP program is a national program that allows high ability students to complete college level work while still in high school. It is sponsored by the College Entrance Examination Board and is available for a variety of courses including calculus and computer science. At the end of the course, students can take exams to qualify for college credit from a large number of colleges and universities. Not all schools offer AP courses, and not all universities accept their test scores for credit. Even many AP classes do not offer a curriculum to match the goals of the NCTM *Standards*.

The International Baccalaureate program is offered in some high schools around the world. The full program is a rigorous one that provides students with a well-rounded education. In the United States it often qualifies graduates for admission to universities with sophomore standing. In addition to mathematics, students must take a foreign language, theory of knowledge, the study of man, experimental studies in the sciences, and one or more options in areas such as art, music, and computer science. Students must also participate in creative, aesthetic, or social service activities and produce an extended essay based on independent research. In mathematics, students may choose to take an examination selected from one of several levels based upon the amount of calculus the student has studied. Many colleges and universities treat an IB certificate in certain courses as equivalent to passing an AP exam.

Universities often offer students the opportunity to receive credit for a course based upon passing an exam even if the students did not take an AP or IB program in high school. Another option at several universities is to have students enrolled concurrently in a university math course and a high school program.

Extracurricular Activities

After School or Saturday Programs. In many areas, enrichment programs are offered after school or on Saturdays. These may be sponsored by the school in the form of a math club or by a local university or a local chapter of a gifted organization. These programs vary by region, and parents or students should check with principals, education departments at local universities, or local chapters of gifted programs to determine the nature of the programs available in the area. If none are available, frequently parents themselves organize such a program. These may charge a nominal fee to pay an instructor and pay for any materials needed such as computer supplies or manipulative math materials. Programs may be open to all motivated, capable students or may have entrance requirements of test scores, teacher and parent recommendations, or an essay written by the student.

Mentorship Programs. Mentorships generally involve placing a student with a person in a career in which the student is interested. These most frequently involve middle or high school students. In mathematics, the mentor might be an accountant, a banking executive, a computer programmer, an engineer, or a person in some other highly technical field. Mentorship programs can be particularly effective with girls and other minorities when they can be placed with a mentor of the same gender or race who can act as a role model. These students often need to see role models in these high level positions because they often believe careers in these areas are reserved for white males. We especially need to encourage girls and students from other traditionally underrepresented groups to consider careers in highly technical fields that involve strong mathematical backgrounds, and mentorships are an effective means of doing this. The mentorship may involve the students working in a summer or after school job with the mentor or may just involve the mentor meeting on an occasional basis with the student to encourage her or him to take those "difficult" math and science courses. Mentors may also come into the classroom to talk to all students about their careers and the necessary prerequisites.

Summer Programs. There are many summer programs across the country that can challenge students in mathematics. These range from regional programs offered at local universities to national programs that draw students from around the country. Some of these are Young Scholars Programs funded by the National Science Foundation.

These are offered to students in junior and senior high school and involve programs in the sciences as well as mathematics programs. These may be either commuter or residential programs and last anywhere from three to nine weeks. Science Service, Inc. publishes a directory of other programs for talented mathematics students, and this address as well as that for the Young Scholars Program can be found in Appendix C. Parents or students can write to them for a listing of programs nationwide.

The Mathematically Precocious Youth program also offers a variety of summer programs across the country. Students must qualify for these programs by scoring well on the SAT or ACT in seventh grade. (See Appendix C for address.)

Many states sponsor statewide programs such as Governor's Schools in the summer. Check with your local State Department of Education for availability in your region. There is also a national network of Governor's Schools programs that includes an annual conference where directors across the United States can share programming ideas.

Competitions. Competitions offer students the opportunity to study math in greater depth, at a higher level, and with a broader curriculum than is offered in most school programs. It also gives students the chance to compete against others with similar abilities. There are many contests at the local, regional, and national levels. Beware of contests that only require quick responses to low level questions. These give students the false impression that mathematics is a thirty-second process requiring only fast recall of memorized facts. The contests that are mentioned in this section are some of the most established, and all require higher level thinking. Contacts for these are listed in Appendix C.

Among the most well-known competitions for secondary students in North America are those sponsored jointly by the National Council of Teachers of Mathematics, the Mathematical Association of America, the Society of Actuaries, Casualty Actuarial Society, Mu Alpha Theta, American Statistical Association, American Mathematical Association of Two-Year Colleges, and the American Mathematical Society. These include the American High School Mathematics Examination (AHSME), the American Junior High School Mathematics Examination (AJHSME), the American Invitational Mathematics Examination (AIME), and the USA Mathematics Olympiad (USAMO). The AHSME is designed with a wide range of difficulty and is for above average students who enjoy mathematics as well as the more exceptional students. It is a 30-item multiple choice test to be taken in 90 minutes using precalculus mathematics. In the 1992-93 school year, nearly 350,000 students took part in this exam. Top scoring students on this exam (approximately the top 1%) are invited to take the AIME, which contains about 15 problems to be solved in 2 1/2 hours in an essay format. About 150 of the top scorers on this exam are invited to the USAMO. The top eight scorers on the USAMO are invited to attend training sessions and to represent the USA in the International Mathematical Olympiad (IMO). The AJHSME is a 25-question, 40-minute exam for students who have not yet completed eighth grade. Over 200,000 students participated in this exam in the 1992-93 school year.

The Mathematical Olympiads for Elementary Schools (MOES) is a team competition for elementary students, usually fifth and sixth graders although students as young as second grade have participated. Teams of up to 35 students participate in five contests held at monthly intervals during the school year. Participants receive certificates, awards, or trophies.

The MathCounts competition is a national mathematics league competition for seventh and eighth graders sponsored by the National Society of Professional Engineers. The coach for both the MathCounts and the Mathematical Olympiad teams is frequently a math teacher and the teams may be part of an after school math club. Coaches are crucial to the success of the programs in terms of motivating students, locating practice materials, and helping students develop their problem solving skills.

One of the few mathematics competitions that involves students as early as kindergarten is the Mathematics Pentathlon. Begun in 1979 in Michigan, the Pentathlon has grown to include over 100,000 Mathletes nationwide. Each division consists of competition involving five mathematics games. Division I is for grades K-1, Division II is for grades 2-3, Division III is for grades 4-5, and Division IV is for grades 6-7. Each division has five games that involve strategies and higher level thinking and frequently use mathematics manipulatives such as Fraction Bars and attribute pieces. Many of the games involve spatial and nonverbal reasoning as well as computation and other mathematical skills. The games are designed so that students with a wide range of skills can compete, but the winners must be able to use critical thinking skills and logical and hypothetical reasoning.

States and regions also frequently sponsor statewide or regional mathematics competitions for students of all ages. Check with your State Department of Education for availability.

Another common local, regional, state, or national competition in mathematics is the math fair. Similar to science fairs, these give students a chance to spend several months developing a mathematics project. One of the largest and most prestigious of these is the Westinghouse Science Talent Search, an annual competition for high school seniors involving research projects in mathematics and engineering as well as other sciences. Students are required to write a one-thousand word report on the project presenting evidence of research ability involving originality. This and the International Science and Engineering Fair are sponsored by Science Service, Inc. and their address is in Appendix C.

Traits of Teachers of Gifted and Talented Mathematics Students

To properly serve the needs of students who exhibit mathematical talents and interest, teachers who understand these students and who engage in deep mathematical thinking themselves are needed. These teachers not only need to be able to diagnose the level of the students but also must be able to suggest appropriate and challenging activities.

Several traits characterize good teachers of gifted and talented mathematics students. Many of these are traits of all good mathematics teachers, but they are especially needed by teachers of our top students. These include:

- an enthusiasm for mathematics and for teaching. Teachers need to be able to convey a sense of the beauty and wonder of mathematics.
- a confidence about their own mathematical abilities. Teachers may not know all the answers to the students' questions, but they should be unafraid to admit a lack of knowledge and to model for the students ways in which to reach an answer on their own.
- a strong mathematical background. In order to challenge students with appropriate problems, teachers must have knowledge of a variety of mathematical topics and should be actively involved in professional development in the field.
- a flexibility and a willingness to be coinvestigators with the students. Students will frequently ask questions that lead the class in directions not foreseen by the teacher. Teachers should be ready and willing to follow the lead of the students as they investigate unplanned areas.
- a willingness to give up the lectern and the chalk. Gifted students need to take over the direction and responsibility for their own learning of mathematics with teachers acting as the "guide on the side."

These characteristics may or may not be present in teachers who are certified to work with gifted students. Frequently, state certification or endorsement for teachers of the gifted and talented does not include preparation in mathematics, especially for elementary teachers. These teachers may have only minimal background in mathematics and may even suffer from math anxiety or math avoidance. As Rogers (1986) stated:

> It is wishful thinking to suppose that hard working teachers without sufficient content knowledge, without special knowledge of gifted children, without time for planning programs, and with limited assistance from supervisory personnel will be able to alter, in any meaningful degree, the educational situation for gifted children. (p. 15)

The need to encourage and develop mathematical talents is a critical one, however, and cannot be ignored. We must give teachers and students the support they need to develop these abilities.

Recommendations

Teachers are perhaps the single most important factor in the development of gifted and talented students. The following recommendations are made for teaching.

1. All teachers should follow the recommendations of the *Professional Standards* and encourage students to construct their own mathematical

understanding, and teachers of the gifted and talented must encourage the highest levels of construction.

2. Teachers must learn to encourage and challenge their top mathematics students. They need adequate resources and support to obtain the materials, technology, and training they need to assist in the development of these students.
3. Students need a variety of rich, challenging mathematics programs from which to choose. They need to experience the joy of solving difficult mathematical problems and should be able to share that joy with others.
4. Parents and teachers should challenge students to ever-increasing levels of mathematical achievement. Teachers need to show students exemplary work from previous students so that students have examples of what can and should be accomplished.

Assessment Standards for School Mathematics

Overview

In October, 1993, NCTM released a working draft of the *Assessment Standards for School Mathematics*. These *Standards* were created to complement the two previous NCTM *Standards*. They were designed to complement, not replace, the Evaluation Standards that were included in the 1989 *Standards*.

The draft of the *Assessment Standards* is based upon the following six assumptions:

- Every student is capable of achieving mathematical power.
- Evidence about student mathematical performance is needed for a variety of purposes.
- For each of these various purposes, information needs to be collected from multiple sources using a variety of methods and formats. There are only three basic sources of such information: observations, student responses to questions, and examinations of student products.
- All evidence about student performance must be considered as a sample of the possible evidence that could have been gathered. As such, there is considerable potential for error when inferences are drawn from the evidence.
- Teachers should be the primary assessors of student performance. No one else in a better position to judge the development of students' mathematical power than a professional teacher who frequently observes, challenges, and listens to students as they investigate problems.
- During their schooling, students should grow in confidence and in their ability to evaluate their own progress and performance. (NCTM, 1993a, p. 3)

Notice that the *Assessment Standards*, like the other two NCTM *Standards,* are based upon the assumption that all students can achieve mathematical power. Again, this does not mean that all students will develop the same mathematical power.

This document contains six assessment standards that represent new criteria for judging the adequacy of mathematics assessment practices and create a basis for building new assessment systems to reflect the reform efforts of NCTM. These are:

- Assessment should reflect the mathematics that is most important for students to learn.
- Assessment should enhance mathematics learning.
- Assessment should promote equity by giving each student optimal opportunities to demonstrate mathematical power and by helping each student meet the profession's high expectations.
- All aspects of the assessment process should be open to review and scrutiny.
- Evidence from assessment activities should yield valid inferences about students' mathematics learning.
- Every aspect of an assessment process should be consistent with the purposes of the assessment. (NCTM, 1993a, p. 27)

The main purpose of assessment should be to gather information about a student's knowledge, abilities, and attitudes toward mathematics and to use this information to plan appropriate learning experiences for each student. For the gifted students, that means using a variety of methods to assess where the student currently stands and then to plan the next steps. Polya's problem solving heuristic can be used here. The problem is to determine the student's current mathematical power, and to then plan to maximize that power. This involves first collecting pertinent data to understand the problem, planning the most appropriate mathematics curriculum and means of instruction, putting the plan into action, and then evaluating the results of that plan. The collection of data and the evaluation of the results require that we use a variety of assessment measures. Some of these that are most appropriate for our top students are described here.

Assessment Strategies

Assessment of mathematical power has in the past frequently been defined as the score on a standardized test of mathematics achievement. Problems with this approach have been discussed previously. Scores on traditional mathematics tests frequently stress lower level skills that a program for talented students should not be emphasizing. In order to properly judge the success of these students, the assessment needs to match the level of thinking being emphasized in the program. Raising the level of thinking in the tests can also serve to raise the level of thinking being taught in the programs. This involves assessment that uses a variety of measures to test the three areas mentioned in the *Assessment Standards*: observations, student responses to questions, and examinations of student products.

Observations

Good classroom teachers learn the most about student performance by watching them in action in the classroom. They may observe students working alone or in groups, and they may or may not ask questions as they observe.

If teachers want to know exactly what a student understands and can express about a mathematical problem, an interview is often helpful. A teacher can ask questions about the student's learning, and can probe to encourage the student to think more deeply about the topic. For our top students, these probing questions are very important. Teachers should ask questions that begin with why, what if, what patterns do you see, and what else does that make you think about, to encourage the student to think about the problem to greater depth and to make connections of greater breadth.

Watching students work in groups can also give a teacher greater insight into each student's understanding. The teacher should notice such things as whether the student seems to take over or sit back, jump right in or reflect, or help the group or hinder it. Even our top students need practice and instruction on how best to work with groups.

Student Responses to Questions

When looking at student responses to questions, it is helpful to have open-ended or open-response questions. Multiple choice or fill-in-the-blank responses frequently give very little information about how the student is thinking. Open-ended questions require that the students explain their reasoning, usually with writing and drawings. The questions may involve computation, but the answers involve a great deal more than finding the correct number. Problems frequently have more than one right answer, and may require that students arrive at a solution in more than one way. Evaluators look for talented students that use novel and creative thinking and can express their reasoning in precise and elegant language. Performance event tasks are similar in that students must express their answers with writing and sketches, but they also involve the use of manipulative materials to solve the problems. Both types of problems require that students apply knowledge to a new situation rather than memorize facts to repeat on a test. Evaluators look for depth of investigations and quality of connections made and not just for a right answer. Students may be required to respond to both open-ended questions and performance event tasks in a set amount of time in a typical classroom setting or problems may be worked upon by students over a longer period of time. Performance event tasks may allow for group problem solving that is typically followed by individual responses to related problems. In these types of problems, it is not difficult for evaluators to recognize outstanding performance in the written responses.

The *New Standards Project,* a joint program of the Learning Research and Development Center at the University of Pittsburgh (Daro, 1993) and the National Center on Education and the Economy is a new program looking at ways of combining teaching, learning and assessment. They are currently piloting a number of interesting problems that students can approach in many different ways that give them a chance to show what they know. Students are encouraged to write about the problems even if they do not finish them, because assessment in this way can focus on what students do know rather than on what they do not know. This project has promise for identifying, developing, and assessing students that have great mathematical power. The address for Philip Daro, Director for Mathematics of the *New Standards Project* is in Appendix C.

Examination of Student Products

Student products may run the gamut from journals and portfolios to oral presentations, math fair projects, multimedia projects, computer programs, and demonstrations.

Journals and portfolios give students a chance to keep a written record of their work in mathematics all year. Work in these may include short responses to open-ended questions and performance event tasks, but students also may take several weeks and even months in working on some of these projects.

Journals generally are a place where students write notes every day. These may be thoughts about work in progress, notes taken about mathematical information, problems that the student is thinking about, or questions that the student wishes to have answered. These are often used by mathematicians, scientists, and engineers in the workplace and students, especially those who are interested in these careers, should get into the habit of keeping them.

Portfolios are a place where students keep some of their best work in mathematics. These generally involve problems that the students have found interesting and have worked on at some length. Talented students should be encouraged to polish these problems, and to go beyond the originally stated problem to find extensions and variations. Teachers should work with students to help them find interesting extensions and variations and to encourage students to explore avenues. Solutions should involve a variety of different methods, problem solving strategies, and tools such as technology and manipulatives. The best of these problems may be expanded for entry into a math fair or other similar competition. Some states now require that students at some grade levels submit a math portfolio for schoolwide accountability purposes. States such as Kentucky have requirements for "Distinguished"-level portfolios that challenge even the very best students. The scoring rubric for Kentucky portfolios is in Appendix D.

Students should not be limited to paper and pencil work in mathematical projects. It is important for them to develop and demonstrate their understanding of mathematics using a variety of materials including video, audio, computer programs, and the construction of concrete models to display their newly constructed mathematics. With today's technology, students can produce and display mathematics far superior to that of even ten years ago.

Standardized Tests

As mentioned in the section on identification, standardized tests have limited use in identifying talented mathematics students and in evaluating programs for them. These may be most useful when out-of-level testing is used that requires higher level thinking. Typical grade level testing does not require enough thinking beyond the memorization or application level to be of much use in a program for gifted students in mathematics. Criterion-referenced tests such as the National Assessment of Educational Progress may also be useful since they do tend to ask questions on a higher level than many achievement tests. Teachers should carefully study the manuals for any tests and a reference on the tests such as the Buros Mental Measurements Yearbook before using them with gifted mathematics students. Be sure the information being tested matches the goals of the program.

Recommendations

To assess mathematical talent and programs designed for gifted and talented students:

1. All teachers should follow the recommendations of the *Assessment Standards* and use a wide variety of assessment measures. The type of assessment used has a profound impact on the type of instruction offered and many standardized achievement tests limit the mathematics to low level computation. It is especially important for teachers of the gifted and talented to expect the highest levels of achievement on several types of assessment.
2. Students should use a wide range of technology and materials to produce quality mathematics. Today's technology allows students to create and display mathematics with outstanding merit.

Summary of Recommendations

Throughout this paper, recommendations have been included following each section. They are repeated here.

Mathematical Talent

In order to allow students to demonstrate that they are capable of top mathematical performance, we must do the following:

1. Give students a wide variety of rich, inviting tasks that require spatial as well as analytic abilities. Both of these are very important to success in mathematics. Good mathematicians must be skilled in both areas.

2. Encourage students to persist in solving mathematical problems. Difficult tasks require work for even the most gifted students. Many students give up before they give mathematics a chance.
3. Expect students to not only solve problems posed by others but to pose and solve new problems of their own. If students are only asked to compute, we will never know who can perform at the top levels in mathematics.

Identification

To identify mathematical talent, the following steps are recommended:

1. We must use a variety of identification measures. Standardized tests measure only a very narrow range of generally low level skills.
2. We must provide students with assessment tasks that tap skills beyond computation. These tasks can frequently not be measured by paper and pencil multiple choice tests.
3. We must have a wide range of opportunities such as exciting mathematics classes, mathematical clubs, and contests where students can demonstrate and hone their mathematical abilities.

Status of Mathematical Talent in the United States

In light of national and international studies of mathematics achievement, the following recommendations can be made:

1. Students in the United States need the opportunity to learn more mathematics. This is especially true of our top students who need the challenge of new and more complex problems, rather than the repetition of a typical mathematics curriculum.
2. Teachers and students in the United States need to tackle fewer problems, but in far greater depth. Time is needed for investigation and discussion, and top students need to be encouraged to delve more deeply into the reasons and connections.
3. We need to encourage students, parents, teachers, and others in our society to believe that all students can learn mathematics and that our top students are capable of greater mathematical power than we have ever asked of them.

Curriculum

Opportunity to learn the curriculum is crucial to the development of gifted and talented students. It should have the following characteristics.

1. All students should follow the core curriculum recommended by the *Standards*. Top students should explore topics in more depth, draw more

generalizations, and create new problems and solutions related to each topic.

2. All students should have access to technology and manipulatives to aid in their construction of mathematical concepts. Top students should use these materials to explore even further.
3. Examples of superior student work should be available to students so they have something to strive for. Olympic athletes would not have progressed as far as they have, if they did not have superior examples of earlier athletes to emulate. The same is needed for student work in mathematics at all levels.

Teaching

Teachers are perhaps the single most important factor in the development of gifted and talented students. The following recommendations are made for teaching.

1. All teachers should follow the recommendations of the *Professional Standards* and encourage students to construct their own mathematical understanding, and teachers of the gifted and talented must encourage the highest levels of construction.
2. Teachers must learn to encourage and challenge their top mathematics students. They need adequate resources and support to obtain the materials, technology, and training they need to assist in the development of these students.
3. Students need a variety of rich, challenging mathematics programs from which to choose. They need to experience the joy of solving difficult mathematical problems and should be able to share that joy with others.
4. Parents and teachers should challenge students to ever-increasing levels of mathematical achievement. Teachers need to show students exemplary work from previous students so that students have examples of what can and should be accomplished.

Assessment

To assess mathematical talent and programs designed for gifted and talented students:

1. All teachers should follow the recommendations of the *Assessment Standards* and use a wide variety of assessment measures. The type of assessment used has a profound impact on the type of instruction offered and many standardized achievement tests limit the mathematics to low level computation. It is especially important for teachers of the gifted and talented to expect the highest levels of achievement on several types of assessment.

2. Students should use a wide range of technology and materials to produce quality mathematics. Today's technology allows students to create and display mathematics with outstanding merit.

Concluding Remarks

Our top students in mathematics are crucial to the well-being of our country. The only way we can meet our national goal of being first in the world in mathematics and science is to raise the mathematical competence of all our students, including the gifted and talented ones.

We must act immediately on a national level to upgrade the level of mathematics being offered to all our top students from kindergarten through graduate school. Perhaps, even more importantly, we must improve the ways in which our students learn mathematics. Teachers must become facilitators of learning to encourage all students to construct new, complex mathematical concepts. Students must be challenged to reach for ever-increasing levels of mathematical understanding. We must strive to help many more students including females, minorities, and students from rural and inner-city schools reach those top levels of mathematical ability. The potential exists in every school in our country for far more expertise in mathematics, and we must help students unlock their talents in this area.

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Appendix A

Draft of the National Council of Teachers of Mathematics Position Paper on Provisions for Mathematically Talented and Gifted Students

Appendix A

Draft of the National Council of Teachers of Mathematics Position Paper on Provisions for Mathematically Talented and Gifted Students

Provisions for Mathematically Talented and Gifted Students

All students, including the mathematically talented and gifted, deserve the opportunity to achieve their full potential. As curricula and instructional practices are developed, school districts must ensure that the mathematically talented and gifted are identified and that their academic needs are considered. Further, it is the responsibility of educators to provide guidance and instruction for such students.

Mathematically talented and gifted students demonstrate an unusually high ability to understand mathematical ideas. These students should be identified through multiple assessment measures. Students themselves, educators, and parents should be involved in this identification process. The evaluators must consider the student's total educational development as well as his or her mathematical ability, achievement, and aspirations.

While all students need curricula that develop the students' problem solving, reasoning, and communication abilities, the mathematically talented and gifted need in-depth and expanded curricula that emphasize higher order thinking skills, nontraditional topics, and the application of skills and concepts in a variety of contexts. The curriculum should provide for all mathematically talented and gifted students every year they are in school. Talented and gifted students should be given a variety of meaningful and productive experiences. Peer-tutoring, acceleration or enrichment, mentorships, separate classes, or programs organized around subject matter areas, flexibility in grouping, and provisions for out-of school activities are all alternatives.

It is the position of NCTM that all students can benefit from an opportunity to study the core curriculum specified in the *Standards*. This can be accomplished by expanding and enriching the curriculum to meet the needs of each individual student, including the gifted and those of lesser capabilities and interests. We challenge teachers and other educators to develop and experiment with course outlines and grouping patterns to present the mathematics in the *Standards* in a meaningful, productive way.
Therefore, the National Council of Teachers of Mathematics recommends that all mathematically talented and gifted students have access to appropriate curricula and instruction that contributes to developing positive attitudes, furthering their mathematical interests, and encouraging their continuing participation in the study of mathematics.

Appendix B

Sources of Additional Materials for Gifted and Talented Mathematics Students

Appendix B

Sources of Additional Materials for Gifted and Talented Mathematics Students

The following is a working list of companies that supply mathematics activity books, journals, manipulatives, and software appropriate for challenging students in mathematics. This list includes only a sample of available materials; teachers will add other favorites of their own. You may write or call to receive information or a free catalog.

Activity Resources Company, Inc.
PO Box 4875
Hayward, CA 94540
1-510-782-1300

Addison-Wesley Publishing Co., Inc.
The Alternative Publishing Group
200 Middlefield Road
Menlo Park, CA 94025
1-800-447-2226

AIMS Education Foundation
PO Box 8120
Fresno, CA 93747-8120
1-209-255-4094

Broderbund Software
PO Box 12947
San Rafael, CA 94913-2937

COMAP, Inc.
Suite 210
57 Bedford Street
Lexington, MA 02173
1-617-862-7878

Creative Publications
5040 West 111th Street
Oak Lawn, IL 60453
1-800-624-0822

Critical Thinking Press and Software
PO Box 448
Pacific Grove, CA 93950
1-800-458-4849

Cuisenaire Co. of America, Inc.
PO Box 5026
White Plains, NY 10602-5026
1-800-237-3142

Dale Seymour Publications
PO Box 10888
Palo Alto, CA 94303-0879
1-800-872-1100

Edmund Scientific Company
101 E. Gloucester Pike
Barrington, NJ 08007-1380
1-609-573-6250

Janson Publications, Inc.
Dept. H-9
PO Box 860
Dedham, MA 02026-0011
1-800-322-MATH

Key Curriculum Press
PO Box 2304
Berkeley, CA 94702
1-800-338-7638

Lawrence Erlbaum Associates, Inc.
365 Broadway
Hillsdale, NJ 07642
1-800-926-6579

Lawrence Hall of Science
University of California
Berkeley, CA 94720
1-510-642-7329

The Learning Company
6493 Kaiser Drive
Fremont, CA 94555
1-800-852-2255

Math Learning Center
PO Box 3226
Salem, OR 97302
1-503-370-8130

Mathematical Association of America
1529 Eighteenth Street, NW
Washington, DC 20036
1-202-387-5200

National Council of Teachers of Mathematics
1906 Association Drive
Reston, VA 22091-1593
1-800-235-7566

Phi Delta Kappa
Center for Dissemination of Innovative Programs
Eighth and Union Street, Box 789
Bloomington, IN 47402

Scholastic Book Services
730 Broadway
New York, NY 10003

School Science and Mathematics
Donald Pratt, Executive Secretary
Department of Curriculum and Foundations
Bloomsburg University
Bloomsburg, PA 17815
1-717-389-4915

Springer-Verlag
175 Fifth Ave.
New York, NY 10010
1-800-777-4643, Ext. 9

Sunburst Communications/Wings for Learning
1600 Green Hills Road
PO Box 660002
Scotts Valley, CA 95067-0002
1-800-321-7511

The Teachers' Laboratory, Inc.
PO Box 6480
Brattleboro, VT 05302-6480
1-802-254-3457

TERC
2067 Massachusetts Ave.
Cambridge, MA 02140
1-617-547-0430

Appendix C

Addresses for Competitions and Other Programs

Appendix C

Addresses for Competitions and Other Programs

The following is a list of addresses for some of the agencies that sponsor competitions and other programs for talented mathematics students:

Advanced Placement Program
College Entrance Examination Board
Princeton, NJ 08540

American Mathematics Competitions (for AHSME, AIME, USAMO, and AJHSME contests)
Dr. Walter E. Mientka
AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0658

International Baccalaureate North America
200 Madison Avenue
New York, NY 10016

MathCounts
National Society of Professional Engineers Information Center
1420 King Street
Alexandria, VA 22314
1-703-684-2828

Mathematical Olympiads for Elementary Schools
George Lenchner
Forest Road School
Valley Stream, NY 11582

Mathematically Precocious Youth Program
Duke University Talent Identification Program
Box 40077
Duke University
Durham, NC 27706-1742
or
Johns Hopkins University
Dr. Luciano Corazza
2701 North Charles Street
Baltimore, MD 21218

Mathematics Pentathlon
PO Box 20590
Indianapolis, IN 46220
1-317-926-MATH

National Council of Teachers of Mathematics
1906 Association Drive
Reston, VA 22091

New Standards Project
Philip Daro, Director for Mathematics
University of California
300 Lakeside Dr.
18th Floor
Oakland, CA 94612-3550
1-510-987-0807

Science Service, Inc.
1719 North Street NW
Washington, DC 20036

U.S.A. Mathematical Talent Search
George Berzsenyi
Department of Mathematics - Box 121
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

Young Scholars Program
National Science Foundation
1800 G Street, NW
Washington, DC 20550

Appendix D

Scoring Rubric for Kentucky's Mathematics Portfolios

Appendix D

Scoring Rubric for Kentucky's Mathematics Portfolios

KENTUCKY MATHEMATICS PORTFOLIO

HOLISTIC SCORING GUIDE

1993-94

An individual portfolio is likely to be characterized by some, but not all, of the descriptors for a particular level. Therefore, the overall score should be the level at which the appropriate descriptors for a portfolio are clustered.

	NOVICE	APPRENTICE	PROFICIENT	DISTINGUISHED
PROBLEM SOLVING Understanding/strategies; Execution/Extensions	• Indicates a basic understanding of problems and uses strategies • Implements strategies with minor mathematical errors in the solution without observations or extensions	• Indicates an understanding of problems and selects appropriate strategies • Accurately implements strategies with solution with limited observations or extensions	• Indicates a broad understanding of problems with alternate strategies • Accurately and efficiently implements and analyzes strategies with correct solution with extensions	• Indicates a comprehensive understanding of problems with efficient, sophisticated strategies • Accurately and efficiently implements and evaluates sophisticated strategies with correct solution and includes analysis, justifications and extensions
REASONING	• Uses mathematical reasoning	• Uses appropriate mathematical reasoning	• Uses perceptive mathematical reasoning	• Uses perceptive, creative and complex mathematical reasoning
MATHEMATICAL COMMUNICATION Language; Representations	• Uses appropriate mathematical language some of the time • Uses few mathematical representations	• Uses appropriate mathematical language • Uses a variety of mathematical representations accurately and appropriately	• Uses precise and appropriate mathematical language most of the time • Uses a wide variety of mathematical representations accurately and appropriately; uses multiple representations within some entries	• Uses sophisticated, precise and appropriate mathematical language throughout • Uses a wide variety of mathematical representations accurately and appropriately; uses multiple representations within entries and states their connections
UNDERSTANDING/CONNECTING CORE CONCEPTS	• Indicates a basic understanding of core concepts	• Indicates an understanding of core concepts with limited connections	• Indicates a broad understanding of some core concepts with connections	• Indicates a comprehensive understanding of core concepts with connections throughout
TYPE AND TOOLS	• Includes few types; uses few tools	• Includes a variety of types; uses tools appropriately	• Includes a wide variety of types; uses a wide variety of tools appropriately	• Includes all types; uses a wide variety of tools appropriately and insightfully

PERFORMANCE DESCRIPTORS	WORKSPACE/ANNOTATIONS
PROBLEM SOLVING • Understands the features of a problem (understands the question, restates the problem in own words) • Explores (draws a diagram, constructs a model and/or chart, records data, looks for patterns) • Selects an appropriate strategy (guesses and checks, makes an exhaustive list, solves a simpler but similar problem, works backward, estimates a solution) • Solves (implements a strategy with an accurate solution) • Reviews, revises, and extends (verifies, explores, analyzes, evaluates strategies/solutions, formulates a rule)	
REASONING • Observes data, records and recognizes patterns, makes mathematical conjectures (inductive reasoning) • Validates mathematical conjectures through logical arguments or counter-examples; constructs valid arguments (deductive reasoning)	
MATHEMATICAL COMMUNICATION • Uses appropriate mathematical notation and terminology • Provides quality explanations for the given task • Expresses concepts, ideas, and reflections clearly • Provides various mathematical representations (models, graphs, charts, diagrams, words, pictures, numerals, symbols, equations)	
UNDERSTANDING/CONNECTING CORE CONCEPTS • Demonstrates an understanding of core concepts • Recognizes, makes, or applies the connections among the mathematical core concepts, to the other disciplines, and to the real world	Place an X on each continuum to indicate the degree of understanding demonstrated for each core concept DEGREE OF UNDERSTANDING OF CORE CONCEPTS (Basic — Comprehensive with Connections) NUMBER MATHEMATICAL PROCEDURES SPACE & DIMENSIONALITY MEASUREMENT CHANGE MATHEMATICAL STRUCTURE DATA

PORTFOLIO CONTENTS

- Table of Contents
- Letter of Reviews
- 5 - 7 Best Entries

BREADTH OF ENTRIES

TYPES
- ◯ INVESTIGATIONS/DISCOVERY
- ◯ APPLICATIONS
- ◯ NON-ROUTINE PROBLEMS
- ◯ PROJECTS
- ◯ INTERDISCIPLINARY
- ◯ WRITING

TOOLS
- ◯ CALCULATORS
- ◯ COMPUTER AND OTHER TECHNOLOGY
- ◯ MODELS/MANIPULATIVES
- ◯ MEASUREMENT INSTRUMENTS
- ◯ OTHERS

◯ GROUP ENTRY

Research-Based Decision Making Series

The National Research Center on the Gifted and Talented

The University of Connecticut
362 Fairfield Road, U-7
Storrs, CT 06269-2007

www.ucc.uconn.edu/~wwwgt

Also of interest
from the

Research-Based Decision Making Series

GROUPING PRACTICES

The Relationship of Grouping Practices to the Education of the Gifted and Talented Learner
Karen B. Rogers

COOPERATIVE Learning

Cooperative Learning and the Academically Talented Student
Ann Robinson

SELF Concept

Self-Concept and the Gifted Child
Robert D. Hoge & Joseph S. Renzulli

ABILITY GROUPING

An Analysis of the Research on Ability Grouping: Historical and Contemporary Perspectives
James A. Kulik

identification in the ARTS

Issues and Practices Related to Identification of Gifted and Talented Students in the Visual Arts
Gilbert A. Clark & Enid Zimmerman

TV & KIDS

Some Children Under Some Conditions: TV and the High Potential Kid
Robert Abelman

Also of interest
from the

Research-Based Decision Making Series

Reading
RESEARCH-BASED DECISION MAKING SERIES

Reading With Young Children
Nancy Jackson & Cathy Roller

EVALUATION
RESEARCH-BASED DECISION MAKING SERIES

Evaluate Yourself
David Fetterman

Creativi

Creativity as an Educational Objective for Disadvantaged Students
Mark A. Runco

Parenting
RESEARCH-BASED DECISION MAKING SERIES

Parenting the Very Young, Gifted Child
Nancy M. Robinson

NRC G/T

The National Research Center on the Gifted and Talented Research Teams

The University of Connecticut

Dr. Francis X. Archambault, Associate Director
The University of Connecticut
School of Education, U-4
Storrs, CT 06269-2064
860-486-4031

Dr. Alexinia Y. Baldwin
Dr. Scott W. Brown
Dr. Deborah E. Burns
Dr. David A. Kenny
Dr. Jonna Kulikowich
Dr. Sally M. Reis
Dr. Karen L. Westberg
Dr. Michael F. Young

The University of Georgia

Dr. Mary M. Frasier, Associate Director
The University of Georgia
Department of Educational Psychology
323 Aderhold Hall
Athens, GA 30602-7146
404-542-5106

Dr. Scott L. Hunsaker

The University of Virginia

Dr. Carolyn M. Callahan, Associate Director
Curry School of Education
The University of Virginia
405 Emmet Street
Charlottesville, VA 22903
804-982-2849

Dr. Michael S. Caldwell
Dr. Marcia A. B. Delcourt
Dr. Brenda H. Loyd
Dr. Kathleen May
Dr. Claudia Sowa
Dr. Ellen Tomchin
Dr. Carol A. Tomlinson

Yale University

Dr. Robert J. Sternberg, Associate Director
Department of Psychology
Yale University
P.O. Box 208205
New Haven, CT 06520-8205
203-432-4632

Dr. Pamela Clinkenbeard